Toeplitz and Hankel algebras - axiomatic and asymptotic aspects

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Abstract

In 1983, the authors introduced a Banach algebra of - as they called them - Toeplitz-like operators. This algebra is defined in an axiomatic way; its elements are distinguished by the existence of four related strong limits. The algebra is in the intersection of Barria and Halmos' asymptotic Toeplitz operators and of Feintuch's asymptotic Hankel operators.

In the present paper, we start with repeating and extending this approach and introduce Toeplitz and Hankel operators in an abstract and axiomatic manner. In particular, we will see that our abstract Toeplitz operators can be characterized both as shift invariant operators and as compressions. Then we show that the classical Toeplitz and Hankel operators on the spaces $H^p(\mathbb{T})$, $l^p(\mathbb{Z}_+)$, and $L^p(\mathbb{R}_+)$ are concrete realizations of our abstract Toeplitz operators. Finally we generalize some results by Didas on derivations on Toeplitz and Hankel algebras to the axiomatic context.

Keywords: Toeplitz-like operators, abstract Toeplitz and Hankel operators, Toeplitz algebras, quasicommutator ideals

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1 Introduction

When an operator A on a Banach space X bears the name *Toeplitz operator* then it is usually distinguished by one of the following properties:

- A is the compression of a "nice" (e.g., a normal) operator onto a non-trivial complementable closed subspace of X, or
- A owns a kind of shift-invariance.

For a concrete example, we recall the definition of the classical Toeplitz operators (as well as of their close relatives, the Hankel operators) on the Hardy space. Let

$1 < p < \infty$. For a function $f \in L^p(\mathbb{T})$, the Lebesgue space over the unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ in the complex plane, we let

$$\hat{f}_n = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\varphi}) e^{-in\varphi} \, d\varphi$$

denote its nth Fourier coefficient. It is well-known that the Riesz projection

$$P : \sum_{n\in\mathbb{Z}} \hat{f}_n e^{in\varphi} \to \sum_{n\in\mathbb{Z}_+} \hat{f}_n e^{in\varphi},$$

is bounded on $L^p(\mathbb{T})$; its range is called the *Hardy space* $H^p = H^p(\mathbb{T})$. We still set $Q := I - P$ and introduce the reflection operator

$$J : L^p(\mathbb{T}) \to L^p(\mathbb{T}), \ (Jf)(t) := t^{-1} f(t^{-1}).$$

Then $J^2 = I$ and $JPJ = Q$.

Let $a \in L^\infty(\mathbb{T})$. We write aI for the operator $f \mapsto af$ of multiplication by a on $L^p(\mathbb{T})$. Then the *Toeplitz operator* $T(a)$ and the *Hankel operator* $H(a)$ on H^p are defined as the compressions

$$T(a) := PaI|_{\mathrm{im\,P}} = PaI|_{H^p} \quad \text{and} \quad H(a) := PaJ|_{\mathrm{imP}} = PaJ|_{H^p},$$

respectively. In particular, $T(a)$ is the compression of the "nice" multiplication operator aI onto the Hardy space. The function a is called the *generating function* of both $T(a)$ and $H(a)$. The assignment $a \mapsto T(a)$ is one-to-one, whereas $a \mapsto H(a)$ is not. Toeplitz and Hankel operators possess completely different properties, but are nevertheless closely related by

$$T(ab) = T(a)T(b) + H(a)H(\tilde{b}), \quad H(ab) = T(a)H(b) + H(a)T(\tilde{b}) \tag{1}$$

where $\tilde{c}(t) := c(1/t)$ for $c \in L^\infty(\mathbb{T})$. Note that $JcJ = \tilde{c}I$.

The function system $\{\chi_n\}_{n\in\mathbb{Z}_+}$, $\chi_n(t) := t^n$ $(t \in \mathbb{T})$, forms a Schauder basis of H^p. The matrix representations of $T(a)$ and $H(a)$, $a \in L^\infty(\mathbb{T})$, with respect to this basis are given by

$$(\hat{a}_{j-k})_{j,k=0}^\infty, \quad (\hat{a}_{j+k+1})_{j,k=1}^\infty,$$

respectively, from which we conclude the other of the distinguishing properties of Toeplitz operators, as follows. For every positive integer n, define

$$V_n : H^p \to H^p, \ f \mapsto \chi_n f \quad \text{and} \quad V_{-n} : H^p \to H^p, \ f \mapsto P(\chi_{-n} f).$$

Then A is a Toeplitz operator on H^p if and only if it is shift-invariant in the sense that

$$A = V_{-n} A V_n \quad \text{for all positive } n \in \mathbb{Z}. \tag{2}$$

This is the starting point for several lines of research. Many work has been done to understand algebras generated by Toeplitz (and Hankel) operators. To mention at least one result in that direction, let $\mathcal{A}$ be a closed subalgebra of $L^\infty(\mathbb{T})$. We denote by $\operatorname{alg} T(\mathcal{A})$ the smallest closed subalgebra of $L(H^p)$, the Banach algebra of the bounded linear operators on H^p, which contains all Toeplitz operators with generating function in $\mathcal{A}$. Further we write $\operatorname{qc}(T)$ for the *quasicommutator ideal* of $\operatorname{alg} T(\mathcal{A})$, i.e., for the smallest closed two-sided ideal of $\operatorname{alg} T(\mathcal{A})$ which contains all operators of the form $T(ab)-T(a)T(b)$ with a, $b \in \mathcal{A}$.

Theorem 1 *(a) The algebra* $\operatorname{alg} T(\mathcal{A})$ *decomposes into the direct sum*

$$\operatorname{alg} \omega(\mathcal{A}) = \omega(\mathcal{A}) \oplus \operatorname{qc}(\omega).$$

(b) The sequence

$$\{0\} \to \operatorname{qc}(T) \to \operatorname{alg} T(\mathcal{A}) \to \mathcal{A} \to \{0\}$$

is short exact; in particular, the quotient algebra $\operatorname{alg} T(\mathcal{A})/\operatorname{qc}(T)$ *is isomorphic to* $\mathcal{A}$*.*

Further results along these lines, also for other kinds of Toeplitz operators, can be found in [4, Section 2.40].

Barria and Halmos [2] took (2) as a starting point to initiate the study of asymptotic Toeplitz operators. They call an operator $A \in L(H^2)$ an *asymptotic Toeplitz operator* if the sequence $(V_{-n}AV_n)_{n\geq 0}$ converges strongly[1]. If the strong limit of that sequence exists, then it is necessarily a Toeplitz operator, whence the notation. They also proved that every operator $A \in \operatorname{alg} TH(L^\infty)$ is asymptotically Toeplitz, where $\operatorname{alg} TH(L^\infty)$ refers to the smallest closed subalgebra of $L(H^2)$ which contains all Toeplitz and Hankel operators with generating functions in L^∞. Since the strong limit of the operators $V_{-n}AV_n$ is a Toeplitz operator $T(\varphi)$ with generating function $\varphi \in L^\infty$, they arrived at a symbol map $\operatorname{smb} : A \mapsto \varphi$. It is proved in [2] that the restriction of smb to $\operatorname{alg} TH(L^\infty)$ is a contractive *-homomorphism from $\operatorname{alg} TH(L^\infty)$ onto L^∞. This symbol map obviously fulfills

$$\operatorname{smb} T(a) = a \quad \text{for every } a \in L^\infty.$$

For the Toeplitz algebra $\operatorname{alg} T(L^\infty) \subset L(H^2)$, this result was already discovered by R. G. Douglas [9] using different methods.

Using a similar strong limit, Feintuch [10] introduced *asymptotic Hankel operators*. Both "asymptotic approaches" have the disadvantage that the sets of all asymptotic Toeplitz (resp. Hankel) operators do not form algebras (see [2], Example 13).

[1] Other kinds of convergence, e.g. norm, weak and Cesaro convergence, are considered as well. Our focus is on strong convergence.

In 1983, we published a preprint [17] where we independently of [2] introduced a class of asymptotic Toeplitz operators (called Toeplitz-like operators in what follows). Besides the strong limit by Barria and Halmos, we required the existence of three other strong limits (one of which is Feintuch's, but note that [10] appeared only in 1989) so that the set of these operators actually forms a Banach algebra. The motivation for introducing that algebra came from the study of the finite section method for operators belonging to the algebra generated by Toeplitz and Hankel operators with piecewise continuous generating functions. In 1985 we published our papers [18], [19] where we presented the results and related applications.

The approach of [17] offers moreover the possibility to introduce Toeplitz and Hankel operators in an abstract and axiomatic manner. Especially, the analogs of (1) and of Theorem 1 can be proved for this class of abstract Toeplitz and Hankel operators. In the present paper, we start with repeating and extending this approach. In particular, we will see that our abstract Toeplitz operators can be characterized both as shift invariant operators and as compressions. Then we show that the classical Toeplitz and Hankel operators on the spaces $H^p(\mathbb{T})$, $l^p(\mathbb{Z}_+)$, and $L^p(\mathbb{R}_+)$ are concrete realizations of our abstract Toeplitz operators. Finally we generalize some results by Didas [7] on derivations on Toeplitz and Hankel algebras to the axiomatic context.

2 Toeplitz-like operators

Here we introduce, in an axiomatic way, the algebra $\mathsf{TL}(X)$ of the Toeplitz-like operators on a Banach space X. For full proofs and some more details we refer to [17, 18] where this algebra was introduced and studied for the first time. A few proofs are sketched here to make the presentation more accessible.

2.1 The axioms

Let X be a Banach space and write $L(X)$ for the Banach algebra of the bounded linear operators on X. The identity operator on X is denoted by I. Further let $G \neq \{0\}$ be a subgroup of the additive group of the real numbers and G_+ the semigroup of its non-negative elements. Our construction of the algebra of the Toeplitz-like is based on two families, $\mathcal{V} = (V_t)_{t\in G}$ and $\mathcal{R} = (R_t)_{t\in G}$, of bounded linear operators on X which are specified and related by the following axioms:

(T_1) The mappings $t \mapsto V_t$ and $t \mapsto V_{-t}$ are semigroup homomorphisms on G_+. In particular, $V_s V_t = V_{s+t}$ and $V_{-s}V_{-t} = V_{-(s+t)}$ for all s, $t \in G_+$.

(T_2) $V_{-t}V_t = I$ but $V_t V_{-t} \neq I$ for all $t \in G_+ \setminus \{0\}$. Thus, for positive t, the operators V_t are invertible only from the left-hand side.

By (T_1) and (T_2), $V_0 = I$. Set $Q_t := V_t V_{-t}$ and $P_t := I - Q_t$ for $t \in G$.

(T_3) $V_t V_{-t} + R_t^2 = I$ and $V_s R_t = R_{s+t} P_t$ for $s,\, t \in G$.

(T_4) $V_{-t} \to 0$ strongly as $t \to \infty$ and $\sup_{t \in G_+}\{\|V_t\|,\, \|R_t\|\} < \infty$.

The following properties are easily verified.

Lemma 2 *(a) $P_t^2 = P_t$ and $Q_t^2 = Q_t$ for $t \in G$.*
(b) $P_{-t} = 0$ and $Q_{-t} = I$ for $t \in G_+$.
(c) $P_t \to I$ and $Q_t \to 0$ strongly as $t \to \infty$.

Thus, P_t and Q_t are complementary projections.

Lemma 3 *For all $s,\, t \in G$,*
(a) $V_s V_t = V_{s+t}$ whenever $s \le 0$ or $t \ge 0$.
(b) $V_s V_t = Q_s V_{s+t} = V_{s+t} Q_{-t}$.
(c) $P_s V_{s+t} = V_{s+t} P_{-t}$.
(d) $P_s P_t = P_{\min\{s,t\}}$ and $Q_s Q_t = Q_{\max\{s,t\}}$.

Lemma 4 *For all $s,\, t \in G$,*
(a) $R_t P_t = P_t R_t = R_t$.
(b) $R_t = 0$ for $t \le 0$.
(c) $R_t V_{-s} = P_t R_{s+t}$.
(d) $R_s R_t = V_{s-t} P_t = P_s V_{s-t}$.

Notice also that

$$\sup_{t \in G}\{\|V_t\|,\, \|P_t\|,\, \|Q_t\|,\, \|R_t\|\} =: M < \infty. \tag{3}$$

We will see several concrete examples of operators V_t and R_t satisfying axioms (T_1) – (T_4) in Section 5. Note also that if the operators V_t and R_t satisfy these axioms on a Banach space X, then the $n \times n$ diagonal matrices

$$\operatorname{diag}(V_t,\, V_t,\, \ldots,\, V_t) \quad \text{and} \quad \operatorname{diag}(R_t,\, R_t,\, \ldots,\, R_t)$$

satisfy these axioms in place of V_t and R_t on the direct sum X^n of n copies of X.

2.2 The algebra $\mathsf{TL}(X)$ of the Toeplitz-like operators

Given families $(V_t)_{t \in G}$ and $(R_t)_{t \in G}$ of operators subject to axioms (T_1) – (T_4), let $\mathsf{TL}(X)$ (with TL for *Toeplitz-like*) stand for the set of all operators $A \in L(X)$ for which the sequences $(V_{-t} A V_t)$, $(R_t A R_t)$, $(V_{-t} A R_t)$ and $(R_t A V_t)$ converge strongly as $t \to \infty$. Their strong limits are denoted by $\mathcal{T}(A)$, $\widetilde{\mathcal{T}}(A)$, $\mathcal{H}(A)$ and $\widetilde{\mathcal{H}}(A)$, respectively.

Example 5 (*a*) The operators V_t and R_t belong to $\mathsf{TL}(X)$, and

$$\mathcal{T}(V_t) = V_t, \quad \widetilde{\mathcal{T}}(V_t) = V_{-t}, \quad \mathcal{H}(V_t) = R_t, \quad \widetilde{\mathcal{H}}(V_t) = 0.$$

The operators R_t lie in the kernel of each of the four strong limits.

(*b*) If $V_t \to 0$ and $R_t \to 0$ weakly as $t \to \infty$, then every compact operator K on X belongs to $\mathsf{TL}(X)$, and

$$\mathcal{T}(K) = \widetilde{\mathcal{T}}(K) = \mathcal{H}(K) = \widetilde{\mathcal{H}}(K) = 0.$$

Theorem 6 (*a*) $\mathsf{TL}(X)$ *is a norm-closed subalgebra of* $L(X)$ *which contains the identity operator.*
(*b*) *The following identities hold for all* A, $B \in \mathsf{TL}(X)$

$$\begin{aligned}
\mathcal{T}(AB) &= \mathcal{T}(A)\mathcal{T}(B) + \mathcal{H}(A)\widetilde{\mathcal{H}}(B), \\
\widetilde{\mathcal{T}}(AB) &= \widetilde{\mathcal{T}}(A)\widetilde{\mathcal{T}}(B) + \widetilde{\mathcal{H}}(A)\mathcal{H}(B), \\
\mathcal{H}(AB) &= \mathcal{H}(A)\widetilde{\mathcal{T}}(B) + \mathcal{T}(A)\mathcal{H}(B), \\
\widetilde{\mathcal{H}}(AB) &= \widetilde{\mathcal{T}}(A)\widetilde{\mathcal{H}}(B) + \widetilde{\mathcal{H}}(A)\mathcal{T}(B).
\end{aligned}$$

Theorem 7 (*a*) $\mathcal{T}$, $\widetilde{\mathcal{T}}$, $\mathcal{H}$ *and* $\widetilde{\mathcal{H}}$ *are bounded linear operators on* $\mathsf{TL}(X)$ *which map this algebra into itself.*
(*b*) *The composition of any two of the operators* $\mathcal{T}$, $\widetilde{\mathcal{T}}$, $\mathcal{H}$ *and* $\widetilde{\mathcal{H}}$ *belongs to* $\{\mathcal{T}, \widetilde{\mathcal{T}}, \mathcal{H}, \widetilde{\mathcal{H}}, 0\}$. *In particular,*

$\circ$	$\mathcal{T}$	$\widetilde{\mathcal{T}}$	$\mathcal{H}$	$\widetilde{\mathcal{H}}$
$\mathcal{T}$	$\mathcal{T}$	$\widetilde{\mathcal{T}}$	0	0
$\widetilde{\mathcal{T}}$	$\widetilde{\mathcal{T}}$	$\mathcal{T}$	0	0
$\mathcal{H}$	$\mathcal{H}$	$\widetilde{\mathcal{H}}$	0	0
$\widetilde{\mathcal{H}}$	$\widetilde{\mathcal{H}}$	$\mathcal{H}$	0	0.

Sketch of the proof. The linearity of the operators in (*a*) is evident; their boundedness is a consequence of (3). So let us verify the first row of the table for example. A basic ingredient are the identities collected in the following lemma.

Lemma 8 *Let* $A \in \mathsf{TL}(X)$ *and* $s \in G_+$. *Then*
(*a*) $V_{-s}\mathcal{T}(A)V_s = \mathcal{T}(A)$,
(*b*) $R_s\mathcal{T}(A)R_s = P_s\widetilde{\mathcal{T}}(A)P_s$,
(*c*) $V_{-s}\mathcal{T}(A)R_s = \mathcal{H}(A)P_s$,
(*d*) $R_s\mathcal{T}(A)V_s = P_s\widetilde{\mathcal{H}}(A)$.

Now let $A \in \mathsf{TL}(X)$ and write

$$\mathcal{T}(A) = V_{-t}AV_t + C_t \quad \text{with } C_t \to 0 \text{ strongly as } t \to \infty.$$

Then, for every fixed $s \in G_+$,

$$V_{-s}\mathcal{T}(A)V_s = V_{-s}V_{-t}AV_tV_s + V_{-s}C_tV_s = V_{-s-t}AV_{s+t} + V_{-s}C_tV_s.$$

Letting t go to ∞ we arrive at identity (a) in Lemma 8, which on its hand implies that $\mathcal{T}(\mathcal{T}(A)) = \mathcal{T}(A)$. In other words, $\mathcal{T} \circ \mathcal{T} = \mathcal{T}$.

Analogously, using (T_3) and Lemma 4 (c) we write

$$R_s\mathcal{T}(A)R_s = R_sV_{-t}AV_tR_s = P_sR_{s+t}AR_{s+t}P_s + R_sC_tR_s,$$

and passage to the strong limit as $t \to \infty$ yields

$$R_s\mathcal{T}(A)R_s = P_s\widetilde{\mathcal{T}}(A)P_s.$$

Letting s go to ∞ we obtain $\widetilde{\mathcal{T}}(\mathcal{T}(A)) = \widetilde{\mathcal{T}}(A)$ or $\widetilde{\mathcal{T}} \circ \mathcal{T} = \widetilde{\mathcal{T}}$. Similarly,

$$V_{-s}\mathcal{T}(A)R_s = V_{-s-t}AR_{s+t}P_s + V_{-s}C_tR_s$$

gives $V_{-s}\mathcal{T}(A)R_s = \mathcal{H}(A)P_s$, whence $\mathcal{H}(\mathcal{T}(A)) = \mathcal{H}(A)$ and, finally,

$$R_sT(A)V_s = P_sR_{s+t}AV_{s+t} + R_sC_tV_s$$

implies $R_s\mathcal{T}(A)V_s = P_s\widetilde{\mathcal{H}}(A)$ and $\widetilde{\mathcal{H}}(\mathcal{T}(A)) = \widetilde{\mathcal{H}}(A)$. The other entries of the table can be checked analogously. ■

Theorem 9 (a) *$\mathcal{T}$ is a bounded projection on $\mathsf{TL}(X)$, and* $\operatorname{im}\mathcal{T}$ *is a closed subspace of $L(X)$.*
(b) $\mathsf{TL}(X) = \operatorname{im}\mathcal{T} \oplus \ker\mathcal{T}$.
(c) $\ker\mathcal{T}$ *is a closed two-sided ideal of $\mathsf{TL}(X)$ and a closed left-sided ideal of $L(X)$.*

It is simple consequence of the last two identities in Theorem 6 that $\ker\mathcal{H}$ and $\ker\widetilde{\mathcal{H}}$ are closed subalgebras of $\mathsf{TL}(X)$.

2.3 Compact operators in $\mathsf{TL}(X)$

As mentioned in Example 5, the compact operators belong to $\mathsf{TL}(X)$ if $V_s \to 0$ and $R_s \to 0$ weakly. We will see now that the converse is also true.

Theorem 10 *The following assertions are equivalent:*
(a) *all operators of rank one are in $\mathsf{TL}(X)$;*
(b) *$V_s \to 0$ and $R_s \to 0$ weakly;*
(c) *all compact operators are in $\mathsf{TL}(X)$.*

Proof. The only implication that needs a proof is $(a) \Rightarrow (b)$. We prepare this proof by a few facts.

Fact 1. *The R_s do not converge strongly.* Indeed, suppose the R_s converge strongly to some operator $R \in L(X)$. Then

$$V_{-s}AR_s \to 0AR = 0 \quad \text{strongly for every } A \in L(X).$$

In particular, $\mathcal{H}(A) = 0$ for every $A \in \mathsf{TL}(X)$. But then $R_s = \mathcal{H}(V_s) = 0$, hence $P_s = R_s^2 = 0$ for every $s \in G_+$, which contradicts the strong convergence of P_s to the identity.

Fact 2. *Every rank one operator in $\mathsf{TL}(X)$ lies in* $\ker \mathcal{T}$. Indeed, let $Kx := \langle x, y \rangle z$ with $y \in X^*$ and $z \in X$. Then

$$V_{-s}KV_s x = \langle V_s x, y \rangle V_{-s} z \to 0$$

since $V_{-s} \to 0$ strongly and $\sup \|V_s\| < \infty$ by axiom (T_4).

Fact 3. *There is a constant $c > 0$ such that*

$$c^{-1}\|P_s x\| \le \|R_s x\| \le c\|P_s x\| \quad \textit{for all } x \in X.$$

Indeed, with $c := \sup \|R_s\|$ we obtain

$$\|R_s x\| = \|R_s P_s x\| \le c\|P_s x\| = c\|R_s^2 x\| \le c^2 \|R_s x\|.$$

Now to the proof of the implication. By Fact 1, there is a $z \in X$ such that the $R_s z$ do not converge. Clearly, $z \neq 0$. Consider the rank one operators

$$K_y x := \langle x, y \rangle z \quad \text{with } y \in X^*.$$

They belong to $\mathsf{TL}(X)$ by assumption and are, hence, in $\ker \mathcal{T}$ by Fact 2. Since $\ker \mathcal{T} = \ker \widetilde{\mathcal{T}}$ by Theorem 7, we conclude that

$$R_s K_y R_s x = \langle R_s x, y \rangle R_s z \to 0 \quad \text{for all } x \in X,\ y \in X^*. \tag{4}$$

Since $\|P_s z\| \to \|z\|$ and by Fact 3, we get

$$\|R_s z\| \ge c^{-1}\|P_s z\| \ge (2c)^{-1}\|z\|$$

for s sufficiently large. Hence, (4) implies that $\langle R_s x, y \rangle \to 0$ for all $x \in X$ and $y \in X^*$; in other words: $R_s \to 0$ weakly.

The proof for V_s runs similarly. Now we use the inclusion $\ker \mathcal{T} \subseteq \ker \widetilde{\mathcal{H}}$ by Theorem 7 to conclude that

$$R_s K_y V_s x = \langle V_s x, y \rangle R_s z \to 0 \quad \text{for all } x \in X,\ y \in X^*$$

from which we obtain the weak convergence $V_s \to 0$ as before. ∎

3 Abstract Toeplitz and Hankel operators

3.1 Symbols

Let $\mathcal{S}mb$ denote the quotient algebra $\mathsf{TL}(X)/\ker\mathcal{T}$, which is correctly defined by Theorem 9, and let smb refer to the canonical homomorphism from $\mathsf{TL}(X)$ onto $\mathcal{S}mb$. We call $\operatorname{smb} A$ the *symbol* of A. The symbol of the identity operator is denoted by e. The symbol of an operator carries important information. For example, we mention the following inverse closedness result from [18], the proof of which makes use of the fact that $\ker\mathcal{T}$ is a left-sided ideal of $L(X)$.

Theorem 11 *Let $A \in \mathsf{TL}(X)$ be invertible in $L(X)$. Then A is invertible in $\mathsf{TL}(X)$ if and only if $\operatorname{smb} A$ is invertible in $\mathcal{S}mb$.*

By Theorem 9 (a), every coset $q = \operatorname{smb} A \in \mathcal{S}mb$ contains exactly one operator from $\operatorname{im}\mathcal{T}$, namely $\mathcal{T}(A)$. We call this operator the *abstract Toeplitz operator* with symbol q and denote it by $\mathsf{T}(q)$. Similarly, we call $\mathsf{H}(q) := \mathcal{H}(\mathsf{T}(q))$ the *abstract Hankel operator* with symbol q. The mappings

$$\mathsf{T} : \mathcal{S}mb \to \operatorname{im}\mathcal{T},\ q \mapsto \mathsf{T}(q) \quad \text{and} \quad \mathsf{H} : \mathcal{S}mb \to \operatorname{im}\mathcal{H},\ q \mapsto \mathsf{H}(q) \tag{5}$$

are linear by construction. Whereas T is also a bijection, H fails to be injective in general. We will see in Corollary 30 that these mappings are bounded.

There is a natural involution $q \mapsto \tilde{q}$ on the symbol algebra $\mathcal{S}mb$ defined by $\tilde{q} := \operatorname{smb} \widetilde{\mathcal{T}}(\mathsf{T}(q))$.

Lemma 12 (a) *The mapping $q \mapsto \tilde{q}$ is an automorphism of $\mathcal{S}mb$ with $\tilde{}\circ\tilde{} = \mathrm{id}$.*
(b) *If $M = 1$ in* (3)*, then $\tilde{}$ is an isometry and $\|q\|_{\mathcal{S}mb} = \|\mathsf{T}(q)\|_{L(X)}$ for all $q \in \mathcal{S}mb$.*

It in not hard to see that $\widetilde{\mathcal{T}}(\mathsf{T}(q)) = \mathsf{T}(\tilde{q})$ and $\widetilde{\mathcal{H}}(\mathsf{T}(q)) = \mathsf{H}(\tilde{q})$. The following lemmas show that abstract Toeplitz and Hankel operators behave as the concrete Toeplitz and Hankel operators introduced in the introduction and deserve, hence, their name.

Lemma 13 *Let p, $q \in \mathcal{S}mb$. Then*

$$\mathsf{T}(pq) = \mathsf{T}(p)\mathsf{T}(q) + \mathsf{H}(p)\mathsf{H}(\tilde{q}), \quad \mathsf{H}(pq) = \mathsf{H}(p)\mathsf{T}(\tilde{q}) + \mathsf{T}(p)\mathsf{H}(q).$$

Proof. Choose A and B in $\mathsf{TL}(X)$ such that $\operatorname{smb} A = p$ and $\operatorname{smb} B = q$. Then $pq = \operatorname{smb}(AB)$, and the operators $\mathsf{T}(p)$, $\mathsf{T}(q)$ and $\mathsf{T}(pq)$ coincide with $\mathcal{T}(A)$, $\mathcal{T}(B)$ and $\mathcal{T}(AB)$, respectively, by definition. Since $A - \mathsf{T}(p)$ and $B - \mathsf{T}(q)$ lie in $\ker\mathcal{T}$, we conclude from Theorem 7 (b) that $\mathcal{H}(A) = \mathcal{H}(\mathsf{T}(p)) = \mathsf{H}(p)$ and $\widetilde{\mathcal{H}}(B) = \widetilde{\mathcal{H}}(\mathsf{T}(q)) = \mathsf{H}(\tilde{q})$. Hence, the assertion of the lemma is a consequence of the identities in Theorem 6 (b). ∎

Lemma 14 *Let $q_1, \ldots, q_n \in \mathcal{S}mb$ and set $A := \prod_{i=1}^n \mathsf{T}(q_i)$ and $q := \prod_{i=1}^n q_i$. Then*

$$\mathcal{T}(A) = \mathsf{T}(q), \quad \widetilde{\mathcal{T}}(A) = \mathsf{T}(\tilde{q}), \quad \mathcal{H}(A) = \mathsf{H}(q), \quad \widetilde{\mathcal{H}}(A) = \mathsf{H}(\tilde{q}).$$

There are several algebras that can be associated with abstract Toeplitz and Hankel operators. The simplest ones are the *Toeplitz algebras* $\operatorname{alg}\mathsf{T}(\mathcal{S})$ where $\mathcal{S}$ is a closed subalgebra of the symbol algebra $\mathcal{S}mb$. By definition, $\operatorname{alg}\mathsf{T}(\mathcal{S})$ is the smallest closed subalgebra of $L(X)$ which contains all abstract Toeplitz operators $\mathsf{T}(c)$ with $c \in \mathcal{S}$. Clearly, $\operatorname{alg}\mathsf{T}(\mathcal{S})$ is a closed subalgebra of $\mathsf{TL}(X)$.

Similarly, given closed subalgebras $\mathcal{S}_1$ and $\mathcal{S}_2$ of the symbol algebra $\mathcal{S}mb$, let $\operatorname{alg}\mathsf{TH}(\mathcal{S}_1, \mathcal{S}_2)$ stand for the smallest closed subalgebra of $L(X)$ which contains all abstract Toeplitz operators $\mathsf{T}(c_1)$ with $c_1 \in \mathcal{S}_1$ and all abstract Hankel operators $\mathsf{H}(c_2)$ with $c_2 \in \mathcal{S}_2$. If $\mathcal{S}_1 = \mathcal{S}_2 =: \mathcal{S}$, then we simply write $\operatorname{alg}\mathsf{TH}(\mathcal{S})$ in place of $\operatorname{alg}\mathsf{TH}(\mathcal{S}, \mathcal{S})$. Algebras of this form are often referred to as *Hankel algebras*. By Theorem 7, $\operatorname{alg}\mathsf{TH}(\mathcal{S}_1, \mathcal{S}_2) \subseteq \mathsf{TL}(X)$. We will see in Section 5.1 that, in general, $\operatorname{alg}\mathsf{TH}(\mathcal{S}mb)$ is properly contained in $\mathsf{TL}(X)$.

A third natural candidate, with a strong coupling between Toeplitz and Hankel operators, is the *Toeplitz-plus-Hankel algebra* $\operatorname{alg}\mathsf{TH}^+(\mathcal{S})$ which is the smallest closed subalgebra of $L(X)$ which contains all Toeplitz-plus-Hankel operators $T(c) + H(c)$ with $c \in \mathcal{S}$. Clearly, $\operatorname{alg}\mathsf{TH}^+(\mathcal{S}) \subseteq \operatorname{alg}\mathsf{TH}(\mathcal{S})$.

3.2 Quasicommutators and decompositions

The goal of this section is to understand the ideal $\ker \mathcal{T}$ as a quasicommutator ideal. A general approach to this circle of ideas is as follows. Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and $D : \mathcal{A} \to \mathcal{B}$ a linear and bounded mapping. We write $\operatorname{alg} D(\mathcal{A})$ for the smallest closed subalgebra of $\mathcal{B}$ which contains all elements $D(a)$ with $a \in \mathcal{A}$. We say that $D(\mathcal{A})$ generates $\mathcal{B}$ (as a Banach algebra) if $\operatorname{alg} D(\mathcal{A}) = \mathcal{B}$.

The *quasicommutator ideal* generated by D is a measure for the deviation of D from being multiplicative (i.e., an algebra homomorphism). This quasicommutator ideal, denoted by $\operatorname{qc}(D)$, is the smallest closed ideal of $\operatorname{alg} D(\mathcal{A})$ which contains all quasi-commutators $D(ab) - D(a)D(b)$ with $a, b \in \mathcal{A}$. Clearly, if D is multiplicative then $\operatorname{qc}(D) = \{0\}$, whereas $\operatorname{qc}(D) = \operatorname{alg} D(\mathcal{A})$ if this algebra is simple and D is not multiplicative. The following lemma provides an equivalent description of $\operatorname{qc}(D)$ is terms of higher quasi-commutators. The proof is an easy exercise.

Lemma 15 *Let $\mathcal{A}, \mathcal{B}$ be Banach algebras and $D : \mathcal{A} \to \mathcal{B}$ a bounded linear mapping such that $D(\mathcal{A})$ generates $\mathcal{B}$. Then*

$$\operatorname{qc}(D) = \operatorname{clos}\operatorname{span}_{\mathcal{B}} \{D(a_1 \ldots a_n) - D(a_1) \ldots D(a_n) : n \in \mathbb{N},\ a_1, \ldots, a_n \in \mathcal{A}\}.$$

In what follows we consider bounded linear mappings $D : \mathcal{A} \to \mathcal{B}$ which own a homomorphic left inverse, i.e. we suppose there is a bounded homomorphism $W : \mathcal{B} \to \mathcal{A}$ such that $W(D(a)) = a$ for all $a \in A$. Mappings D with that property are often called *discretizations* or *quantizations*, in order to emphasize their role in numerical analysis or operator theory, respectively (see, e.g., Section 6.5.3 in [12]).

Theorem 16 *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras, $D : \mathcal{A} \to \mathcal{B}$ a bounded linear mapping, and $W : \mathcal{B} \to \mathcal{A}$ a bounded homomorphism such that $W(D(a)) = a$ for every $a \in \mathcal{A}$. Then*

(a) D is injective, $D(\mathcal{A})$ is a closed subspace of $\mathcal{B}$, and W is surjective,

(b) the algebra $\mathcal{B}$ decomposes into the direct sum

$$\mathcal{B} = D(\mathcal{A}) \oplus \ker W, \tag{6}$$

(c) if $D(\mathcal{A})$ generates $\mathcal{B}$, then $\ker W = \mathrm{qc}\,(D)$.

(d) if D and W are contractions, then D is an isometry, the canonical projection $P : \mathcal{B} \to D(\mathcal{A})$ associated with the decomposition $\mathcal{B} = D(\mathcal{A}) \oplus \ker W$ has norm 1, and

$$\|D(a)\| = \min_{k \in \ker W} \|D(a) + k\| \quad \text{for every } a \in \mathcal{A}. \tag{7}$$

Proof. (a) First we show that $D(\mathcal{A})$ is a closed subspace of $\mathcal{B}$. Let $D(a_n)$ be a sequence in $D(\mathcal{A})$ which converges to $b \in B$. Then the $W(D(a_n)) = a_n$ form a Cauchy sequence in $\mathcal{A}$ which converges to an element $a \in \mathcal{A}$. Since D is bounded, the $D(a_n)$ converge to $D(a)$; hence $b = D(a) \in D(\mathcal{A})$, and $D(\mathcal{A})$ is closed. The other assertions follow immediately from $W \circ D = \mathrm{id}_{\mathcal{A}}$.

(b) Writing $b \in \mathcal{B}$ as $b = D(W(b)) + (b - D(W(b))$ and noting that

$$W(b - D(W(b)) = W(b) - W(D(W(b))) = 0$$

we obtain $\mathcal{B} = D(\mathcal{A}) + \ker W$. To see that this sum is direct, assume that $D(a) \in \ker W$. Then $a = W(D(a)) = 0$, hence $D(a) = 0$.

(c) Since $W(D(ab) - D(a)D(b)) = W(D(ab)) - W(D(a))W(D(b)) = 0$ and $\mathrm{qc}\,(D)$ is a closed ideal of $\mathcal{B}$, the inclusion $\mathrm{qc}\,(D) \subseteq \ker W$ is clear (and it holds without the additional assumption in (c)).

For the reverse inclusion, let $k \in \ker W$. Since $D(\mathcal{A})$ generates $\mathcal{B}$, there is a sequence of elements $k_n = \sum_i \prod_j D(a_{ijn})$ with $a_{ijn} \in \mathcal{A}$ such that $\|k - k_n\| \to 0$. Using Lemma 15, we can write k_n as

$$k_n = D(\sum_i \prod_j a_{ijn}) + q_n = D(W(k_n)) + q_n = D(W(k_n - k)) + q_n$$

with $q_n \in \mathrm{qc}\,(D)$ (note that $W(k) = 0$). Then

$$\begin{aligned} \|k - q_n\| &\le \|k - k_n\| + \|k_n - q_n\| \le \|k - k_n\| + \|D(W(k_n - k))\| \\ &\le (1 + \|D\| \, \|W\|) \, \|k - k_n\| \to 0. \end{aligned}$$

Thus, k can be approximated as closely as desired by elements in $\mathrm{qc}\,(D)$. Since the quasi-commutator ideal is closed, the assertion follows.

(d) Let $a \in \mathcal{A}$ and $k \in \ker W$. Then $\|a\| = \|W(D(a))\| \leq \|D(a)\| \leq \|a\|$ and

$$\|D(a)\| = \|D(W(D(a)))\| = \|D(W(D(a)+k))\| \leq \|D(a)+k\|,$$

which implies assertion (d). ∎

The conditions in Theorem 16 are also necessary, as the following proposition shows.

Proposition 17 *Let the Banach algebra $\mathcal{B} = \mathcal{L} \oplus \mathcal{J}$ be the direct sum of a closed subspace $\mathcal{L}$ and a closed ideal $\mathcal{J}$. Then there are a Banach algebra $\mathcal{A}$, a bounded linear mapping $D : \mathcal{A} \to \mathcal{L}$ such that $D(\mathcal{A}) = \mathcal{L}$, and a bounded homomorphism $W : \mathcal{B} \to \mathcal{A}$ which is a left inverse for D. Moreover, $\mathcal{J} = \ker W$, and $\mathcal{J} = \ker W = \mathrm{qc}\,(D)$ if $\mathcal{L}$ generates $\mathcal{B}$.*

Proof. Let $P : \mathcal{B} \to \mathcal{L}$ be the canonical projection associated with the decomposition $\mathcal{B} = \mathcal{L} \oplus \mathcal{J}$, set $\mathcal{A} := \mathcal{B}/\mathcal{J}$, and write W for the canonical homomorphism from $\mathcal{B}$ onto $\mathcal{A}$. Then the mapping $D : \mathcal{A} \to \mathcal{L}, a + \mathcal{J} \mapsto P(a)$, is correctly defined, linear and surjective, and

$$W \circ D : a + \mathcal{J} \to P(a) \to P(a) + \mathcal{J} = a + \mathcal{J}$$

is the identity mapping on $\mathcal{A}$. Since $\mathcal{L}$ is closed and D is inverse to the (evidently bounded and surjective) mapping $W|_{\mathcal{L}} : \mathcal{L} \to \mathcal{A}$, D is bounded by the open mapping theorem. The equality $\mathcal{J} = \ker W = \mathrm{qc}\,(D)$ is a consequence of Theorem 16. ∎

3.3 Algebras of abstract Toeplitz and Hankel operators

Next we are going to apply these general results in the context of (abstract) Toeplitz and Hankel algebras. We start with the Toeplitz algebra $\mathrm{alg}\,\mathsf{T}(\mathcal{S})$, with $\mathcal{S}$ a closed subalgebra of $\mathcal{S}mb$. Slightly modifying the above notation, we write $\mathrm{qc}_T\,(\mathcal{S})$ (instead of $\mathrm{qc}\,(\mathsf{T}|_{\mathcal{S}})$) for the quasicommutator ideal of the mapping $\mathsf{T} : \mathcal{S}mb \to L(X)$ restricted to $\mathcal{S}$. Thus, $\mathrm{qc}_T\,(\mathcal{S})$ is the smallest closed ideal of $\mathrm{alg}\,\mathsf{T}(\mathcal{S})$ which contains all operators $\mathsf{T}(cd) - \mathsf{T}(c)\mathsf{T}(d)$ with $c, d \in \mathcal{S}$. Then the restriction of the symbol homomorphism smb to $\mathrm{alg}\,\mathsf{T}(\mathcal{S})$ is a left inverse to the linear mapping T, restricted to $\mathcal{S}$. In this context, Theorem 16 specifies as follows.

Corollary 18 (a) $\mathrm{qc}_T\,(\mathcal{S}) = \ker(\mathrm{smb}\,|_{\mathrm{alg}\,\mathsf{T}(\mathcal{S})}) = \ker(\mathcal{T}|_{\mathrm{alg}\,\mathsf{T}(\mathcal{S})})$.
(b) *The algebra* $\mathrm{alg}\,\mathsf{T}(\mathcal{S})$ *decomposes into the direct sum*

$$\mathrm{alg}\,\mathsf{T}(\mathcal{S}) = \mathsf{T}(\mathcal{S}) \oplus \mathrm{qc}_T\,(\mathcal{S}).$$

(c) The sequence

$$\{0\} \to \mathrm{qc}_T\,(\mathcal{S}) \to \operatorname{alg} \mathsf{T}(\mathcal{S}) \to \mathcal{S} \to \{0\},$$

with the second arrow standing for the natural embedding and the third one for the restriction of the symbol mapping, is exact.

In general, the Hankel algebras $\operatorname{alg} \mathsf{TH}(\mathcal{S}_1, \mathcal{S}_2)$ do not arise from a discretization mapping in the above sense; in particular there is no natural quasicommutator ideal. So we deal with the kernel of $\mathcal{T}$ instead of the quasicommutator ideal. Theorem 9 specifies as follows to this context.

Corollary 19 *(a)* $\operatorname{alg} \mathsf{TH}(\mathcal{S}_1, \mathcal{S}_2) = \mathsf{T}(\mathcal{S}_1) \oplus \ker(\mathcal{T}|_{\operatorname{alg} \mathsf{TH}(\mathcal{S}_1, \mathcal{S}_2)})$.
(b) The sequence

$$\{0\} \to \ker(\mathcal{T}|_{\operatorname{alg} \mathsf{TH}(\mathcal{S}_1, \mathcal{S}_2)}) \to \operatorname{alg} \mathsf{TH}(\mathcal{S}_1, \mathcal{S}_2) \to \mathcal{S}_1 \to \{0\},$$

where the second arrow stands for the embedding and the third one for the restriction of the symbol mapping, is exact.

In this context, there is an equivalent description of the kernel of $\mathcal{T}$ (see [7] for the H^2-setting).

Lemma 20 *Let* $\mathcal{S}_1 \subseteq \mathcal{S}_2$. *Then* $\ker(\mathcal{T}|_{\operatorname{alg} \mathsf{TH}(\mathcal{S}_1, \mathcal{S}_2)})$ *is the smallest closed ideal of* $\operatorname{alg} \mathsf{TH}(\mathcal{S}_1, \mathcal{S}_2)$ *which contains all Hankel operators* $\mathsf{H}(q)$ *with* $q \in \mathcal{S}_2$.

Proof. Abbreviate $\ker(\mathcal{T}|_{\operatorname{alg} \mathsf{TH}(\mathcal{S}_1, \mathcal{S}_2)})$ by $\mathcal{J}_1$ and the closed ideal generated by $\mathsf{H}(\mathcal{S}_2)$ by $\mathcal{J}_2$. Since every Hankel operator is in $\ker \mathcal{T}$, it is immediate that $\mathcal{J}_2 \subseteq \mathcal{J}_1$. For the reverse inclusion, let $A \in \mathcal{J}_1$. Then A can be represented as a norm limit

$$A = \lim_{n \to \infty} \left(\sum_{i=1}^{\alpha_n} A_i^{(n)} \mathsf{H}(b_i^{(n)}) C_i^{(n)} + \sum_{i=1}^{\gamma_n} \prod_{j=1}^{\delta_n} \mathsf{T}(d_{ij}^{(n)}) \right)$$

with symbols $d_{ij}^{(n)} \in \mathcal{S}_1$ and $b_i^{(n)} \in \mathcal{S}_2$ and with operators $A_i^{(n)}, C_i^{(n)} \in \mathsf{TH}(\mathcal{S}_1, \mathcal{S}_2)$. Since both A and all Hankel operators are in $\ker \mathcal{T}$ and $\mathcal{T}$ is continuous, we conclude from Lemma 14 that

$$0 = \mathcal{T}(A) = \lim_{n \to \infty} \mathsf{T} \left(\sum_{i=1}^{\gamma_n} \prod_{j=1}^{\delta_n} d_{ij}^{(n)} \right).$$

Hence, $A = A - \mathcal{T}(A)$ is equal to

$$\lim_{n \to \infty} \left(\sum_{i=1}^{\alpha_n} A_i^{(n)} \mathsf{H}(b_i^{(n)}) C_i^{(n)} + \sum_{i=1}^{\gamma_n} \prod_{j=1}^{\delta_n} \mathsf{T}(d_{ij}^{(n)}) - \mathsf{T} \left(\sum_{i=1}^{\gamma_n} \prod_{j=1}^{\delta_n} d_{ij}^{(n)} \right) \right).$$

The first item is evidently in $\mathcal{J}_2$; the second one is in the quasicommutator ideal of $\mathsf{T}(\mathcal{S}_1)$ by Lemma 15. This ideal is generated by the quasicommutators

$$\mathsf{T}(a)\mathsf{T}(b) - \mathsf{T}(ab) = -\mathsf{H}(a)\mathsf{H}(\tilde{b})$$

which are in $\mathcal{J}_2$ because of $\mathcal{S}_1 \subseteq \mathcal{S}_2$. ■

In contrast to the algebras $\operatorname{alg}\mathsf{TH}(\mathcal{S}_1, \mathcal{S}_2)$, the Toeplitz-plus-Hankel algebras $\operatorname{alg}\mathsf{TH}^+(\mathcal{S})$ again arise from a discretization mapping, namely from the restriction of the mapping

$$\mathsf{T} + \mathsf{H} : \mathcal{S}mb \to \mathsf{LT}(X), \quad s \mapsto \mathsf{T}(s) + \mathsf{H}(s)$$

to $\mathcal{S}$, again with smb as a homomorphic left inverse. We denote the related quasicommutator ideal by $\operatorname{qc}_{T+H}(\mathcal{S})$, i.e. $\operatorname{qc}_{T+H}(\mathcal{S})$ is the smallest closed ideal of $\operatorname{alg}\mathsf{TH}^+(\mathcal{S})$ which contains all operators

$$\mathsf{T}(cd) + \mathsf{H}(cd) - (\mathsf{T}(c) + \mathsf{H}(c))(\mathsf{T}(d) + \mathsf{H}(d)) \quad \text{with } c,\, d \in \mathcal{S}.$$

Corollary 21 (a) $\operatorname{qc}_{T+H}(\mathcal{S}) = \ker(\operatorname{smb}|_{\operatorname{alg}\mathsf{TH}^+(\mathcal{S})}) = \ker(\mathcal{T}|_{\operatorname{alg}\mathsf{TH}^+(\mathcal{S})})$.
(b) *The algebra* $\operatorname{alg}\mathsf{TH}^+(\mathcal{S})$ *decomposes into the direct sum*

$$\operatorname{alg}\mathsf{TH}^+(\mathcal{S}) = \{\mathsf{T}(c) + \mathsf{H}(c) : c \in \mathcal{S}\} \oplus \operatorname{qc}_{T+H}(\mathcal{S}).$$

(c) *The sequence*

$$\{0\} \to \operatorname{qc}_{T+H}(\mathcal{S}) \to \operatorname{alg}\mathsf{TH}^+(\mathcal{S}) \to \mathcal{S} \to \{0\},$$

with the second arrow standing for the natural embedding and the third one for the restriction of the symbol mapping, is exact.

We conclude by a few remarks on commutators. The *commutator ideal* $\operatorname{com}\mathcal{B}$ of a Banach algebra $\mathcal{B}$ is the smallest closed ideal of $\mathcal{B}$ which contains all commutators $ab - ba$ with $a,\, b \in \mathcal{B}$. It is easy to see that in the context of Theorem 16 and if $D(\mathcal{A})$ generates $\mathcal{B}$, the commutator ideal of $\mathcal{B}$ is already generated by commutators $D(c)D(d) - D(d)D(c)$ with $c,\, d \in \mathcal{A}$. Thus, if $\mathcal{A}$ is commutative, then $\operatorname{com}\mathcal{B}$ is contained in $\operatorname{qc}(\mathcal{A})$, which follows from

$$D(c)D(d) - D(d)D(c) = (D(c)D(d) - D(cd)) - (D(d)D(c) - D(dc)).$$

Note also that the commutator ideal of a closed subalgebra $\mathcal{A}$ of $\mathsf{LT}(X)$ with commutative symbol algebra $\operatorname{smb}\mathcal{A}$ is contained in $\ker\mathcal{T}$.

The above remarks apply in particular to Toeplitz and Toeplitz-plus-Hankel algebras. For the commutator ideal of Hankel algebras, we have the following.

Lemma 22 *If* $\mathcal{S}_1$ *is commutative, then* $\operatorname{com}\operatorname{alg}\mathsf{TH}(\mathcal{S}_1, \mathcal{S}_2)$ *is contained in* $\ker\mathcal{T}$.

Indeed, let $A, B \in \operatorname{alg} TH(\mathcal{S}_1, \mathcal{S}_2)$. By Corollary 19, we write $A = \mathsf{T}(a) + K$ and $B = \mathsf{T}(b) + L$ with $a, b \in \mathcal{S}_1$ and $K; L \in \ker \mathcal{T}$. Then

$$AB - BA = \mathsf{T}(a)\mathsf{T}(b) - \mathsf{T}(b)\mathsf{T}(a) + M$$

with $M \in \ker \mathcal{T}$. The assertion follows since

$$\mathcal{T}(\mathsf{T}(a)\mathsf{T}(b) - \mathsf{T}(b)\mathsf{T}(a)) = \mathsf{T}(ab - ba) = 0$$

by Lemma 14 and since $\mathcal{S}_1$ is commutative. ∎

3.4 Continuous symbols

The operators V_t belong to the algebra $\mathsf{TL}(X)$ as we observed in Example 5. Let $\operatorname{alg}(\mathcal{V})$ stand for the smallest closed subalgebra of $\mathsf{TL}(X)$ which contains all of these operators.

Lemma 23 *Let $\mathcal{A}$ be a closed subalgebra of $\mathsf{TL}(X)$ which contains $\mathcal{V}$. Then the following sets are equal for every $t_0 \in G_+ \setminus \{0\}$:*
(a) the smallest closed ideal of $\mathcal{A}$ which contains all operators P_t, $t \in G$;
(b) the smallest closed ideal of $\mathcal{A}$ which contains P_{t_0}.

Proof. Let $\mathcal{I}$ stand for the smallest closed ideal of $\mathcal{A}$ which contains P_{t_0}. Clearly, $\mathcal{I}$ is contained in the ideal described in (a). For the reverse inclusion, we have to show that $P_t \in \mathcal{I}$ for every $t \in G_+$. Let $k \in \mathbb{Z}_+$ and $k \geq 2$. Then

$$\begin{aligned} P_{kt_0} - P_{(k-1)t_0} &= V_{(k-1)t_0} V_{-(k-1)t_0} - V_{kt_0} V_{-kt_0} \\ &= V_{(k-1)t_0}(I - V_{t_0} V_{-t_0}) V_{-(k-1)t_0} \\ &= V_{(k-1)t_0} P_{t_0} V_{-(k-1)t_0} \in \mathcal{I}. \end{aligned}$$

Summing up we conclude that $P_{kt_0} \in \mathcal{I}$ for every $k \in \mathbb{Z}_+$. Now, given $t \in G_+$, choose $k \in \mathbb{Z}_+$ such that $t < kt_0$. Then, by Lemma 3 (a),

$$\begin{aligned} V_{-(kt_0-t)} P_{kt_0} V_{kt_0-t} &= I - V_{-(kt_0-t)} V_{kt_0} V_{-kt_0} V_{kt_0-t} \\ &= I - V_t V_{-t} = P_t, \end{aligned}$$

whence $P_t \in \mathcal{I}$. ∎

In particular we see that if one of the projections P_t with $t \in G_+ \setminus \{0\}$ is compact, then each of these projections is compact.

Let $\mathcal{I}(\mathcal{P})$ denote the smallest closed ideal of $\operatorname{alg}(\mathcal{V})$ which contains (one or all of) the projections P_t. Further write φ_t for the symbol of V_t and $\mathcal{C}$ for the smallest closed subalgebra of the symbol algebra $\mathcal{S}mb$ which contains all symbols φ_t. We call $\mathcal{C}$ the *algebra of the continuous symbols*. This notion is inspired by Gohberg and Feldman's text [11]. Note that $\varphi_s \varphi_t = \varphi_{s+t}$ for all $s, t \in G$ and that φ_0 is the identity element of $\mathcal{S}mb$.

Lemma 24 *(a)* $\operatorname{alg}(\mathcal{V}) = \operatorname{alg}(\mathsf{T}(\mathcal{C}))$.
(b) $\mathcal{I}(\mathcal{P}) = \ker(\mathcal{T}|_{\operatorname{alg}(\mathcal{V})}) = \operatorname{qc}_T(\mathcal{C})$.

Proof. Assertion (a) is a consequence of $\mathsf{T}(\varphi_t) = V_t$. The second equality in (b) comes from Corollary 18. The first equality is then evident since the quasicommutators $P_t = I - V_t V_{-t}$ belong to the kernel of $\mathcal{T}$, and they generate the quasicommutator ideal $\operatorname{qc}_T(\mathcal{C})$. ■

Whereas the symbol algebra $\mathcal{S}mb$ need not to be commutative, its subalgebra of the continuous symbols is commutative. Moreover, the following holds.

Lemma 25 *$\mathcal{C}$ is a closed subalgebra in the center of $\mathcal{S}mb$.*

Proof. Let $q \in \mathcal{S}mb$ and $t \in G_+$. By Lemma 8 (a), $V_{-t}\mathsf{T}(q)V_t = \mathsf{T}(q)$. Passing to symbols we obtain $\varphi_{-t} q \varphi_t = q$ or, equivalently, $q\varphi_t = \varphi_t q$ for $t \in G_+$. Since $\varphi_{-t} = (\varphi_t)^{-1}$, this equality holds for negative t as well. ■

3.5 Abstract Toeplitz operators as compressions

We infer from Lemma 8 (a) that the elements of $\operatorname{im}\mathcal{T}$, hence all abstract Toeplitz operators, are shift invariant. We will see now that abstract Toeplitz operators also arise as compressions of abstract Laurent operators, for which we have to embed X into a larger space. This construction will also shed new light onto the algebra $\mathsf{TL}(X)$ since it identifies the four strong limits $\mathcal{T}$, $\widetilde{\mathcal{T}}$, $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ as components of a single strong limit.

Let the Banach space X, the group G and the operators V_t, R_t, P_t and Q_t with $t \in G$ be as in Section 2.1. Let X^2 stand for the direct sum $X \oplus X$, provided with the norm $\|(x_1, x_2)\| = \|x_1\|_X + \|x_2\|_X$ (or with another norm making X^2 to a Banach space into which X is isometrically embedded). Thinking of the elements of X^2 as column vectors, we identify operators in $L(X^2)$ with 2×2 matrices with entries in $L(X)$.

For $t \in G_+$, define operators $U_{\pm t}$ on X^2 by

$$U_t := \begin{pmatrix} V_t & R_t \\ 0 & V_{-t} \end{pmatrix} \quad \text{and} \quad U_{-t} := \begin{pmatrix} V_{-t} & 0 \\ R_t & V_t \end{pmatrix}. \tag{8}$$

Proposition 26 *The family $\{U_t\}_{t \in G}$ forms a commutative group, i.e. $U_s U_t = U_t U_s = U_{s+t}$ for all $s, t \in G$.*

Proof. We prepare the proof by establishing the identities

$$V_s V_t + R_s R_{-t} = V_{s+t} \quad \text{for all } s, t \in G \tag{9}$$

$$V_s R_t + R_s V_{-t} = R_{s+t} \quad \text{for all } s, t \in G. \tag{10}$$

From Lemmas 3 (b) and 4 (d) we know that $R_s R_{-t} = P_s V_{s+t}$ and $V_s V_t = Q_s V_{s+t}$. Summing up gives (9). For (10), recall from Axiom (T_3) and Lemma 4 (c) that

$V_sR_t = R_{s+t}P_t$ and $R_sV_{-t} = P_sR_{s+t}$. Since $V_{-t} = V_{-t}Q_t$, the last identity implies that $R_sV_{-t} = P_sR_{s+t}Q_t$. It thus remains to show that

$$P_sR_{s+t}Q_t = R_{s+t}Q_t \quad \text{or, equivalently,} \quad Q_sR_{s+t}Q_t = 0,$$

which is further equivalent to $V_{-s}R_{s+t}V_t = 0$. Now we have

$$\begin{array}{rcll} V_{-s}R_{s+t}V_t & = & R_tP_{s+t}V_t & \text{by Axiom } (T_3) \\ & = & R_tP_tP_{s+t}V_t & \text{by Lemma 4}\,(a) \\ & = & R_tP_{s+t}P_tV_t & \text{by Lemma 3}\,(d) \\ & = & 0 & \text{since } P_tV_t = 0. \end{array}$$

The identities (9) and (10) together imply that

$$\begin{pmatrix} V_s & R_s \\ R_{-s} & V_{-s} \end{pmatrix} \begin{pmatrix} V_t & R_t \\ R_{-t} & V_{-t} \end{pmatrix} = \begin{pmatrix} V_{s+t} & R_{s+t} \\ R_{-s-t} & V_{-s-t} \end{pmatrix}$$

for all $s, t \in G$, from which the assertion follows (note that $R_t = 0$ for $t \leq 0$). ■

The simple identity

$$U_{-t} \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} U_t = \begin{pmatrix} V_{-t}AV_t & V_{-t}AR_t \\ R_tAV_t & R_tAR_t \end{pmatrix},$$

holding for every $A \in L(X)$ and $t \in G_+$, implies the relation between the two approaches to Toeplitz-like operators.

Theorem 27 *Let $A \in L(X)$. Then the strong limit*

$$\mathcal{L}(A) := \text{s-lim}_{t\to\infty} U_{-t} \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} U_t$$

exists if and only if $A \in \mathsf{TL}(X)$. In this case,

$$\mathcal{L}(A) = \begin{pmatrix} \mathcal{T}(A) & \mathcal{H}(A) \\ \widetilde{\mathcal{H}}(A) & \widetilde{\mathcal{T}}(A) \end{pmatrix}. \tag{11}$$

Corollary 28 (a) *The mapping $\mathcal{L} : \mathsf{TL}(X) \to L(X^2)$ is a bounded homomorphism.*
(b) *$U_{-t}\mathcal{L}(A)U_t = \mathcal{L}(A)$ for every $A \in \mathsf{TL}(X)$ and $t \in G$.*

Assertion (a) comes from the definition of $\mathcal{L}$ and Theorem 6 (b); assertion (b) can be proved in the same as Lemma 8 (likewise: it follows from that lemma). ■

Define mappings

$$E : X \to X^2,\ x \mapsto (x,\, 0)^T, \qquad E^{-1} : X^2 \to X,\ (x,\, y)^T \mapsto x.$$

Then, by (11),

$$E^{-1}\mathcal{L}(A)E = \mathcal{T}(A) \qquad \text{for every } A \in \mathsf{LT}(X). \tag{12}$$

In other words, if $P \in L(X^2)$ stands for the projection $(x, y)^T \mapsto (x, 0)^T$, then every abstract Toeplitz operator can be viewed as the compression $PA|_{\mathrm{imP}}$ of an *abstract Laurent operator* $A \in \mathrm{im}\,\mathcal{L}$.

Theorem 29 *(a)* $\mathrm{im}\,\mathcal{L}$ *is a closed subalgebra of* $L(X^2)$.
(b) The mapping $A \mapsto \mathrm{smb}\,(E^{-1}AE)$ *is a bounded isomorphism from* $\mathrm{im}\,\mathcal{L}$ *onto* $\mathcal{S}mb = \mathrm{smb}\,\mathsf{TL}(X)$.

Proof. *(a)* Let $\mathcal{L}(A_n)$ be a sequence in $\mathrm{im}\,\mathcal{L}$ which converges in $L(X^2)$. Then, by (12), $(\mathcal{T}(A_n))$ is a Cauchy sequence in $\mathrm{im}\,\mathcal{T}$. Since $\mathrm{im}\,\mathcal{T}$ is closed by Theorem 9 *(a)*, there is a operator $A \in \mathsf{LT}(X)$ such that $\mathcal{T}(A_n) \to \mathcal{T}(A)$ in the operator norm. The first column in the table in Theorem 7 shows that $\mathcal{L}(\mathcal{T}(B)) = \mathcal{L}(B)$ for every $B \in \mathsf{TL}(X)$. Hence, $\mathcal{L}(A_n) = \mathcal{L}(\mathcal{T}(A_n))$ converges in the norm to $\mathcal{L}(A) = \mathcal{L}(\mathcal{T}(A))$. Thus, $\mathrm{im}\,\mathcal{L}$ is a closed subspace of $L(X^2)$. It is also an algebra by Corollary 28.

(b) The mapping $A \mapsto \mathrm{smb}\,(E^{-1}AE)$ sends $\mathcal{L}(A)$ to $\mathrm{smb}\,\mathcal{T}(A)$ by (12). This mapping is a homomorphism because $\mathcal{T}(A) - \mathcal{T}(A)\mathcal{T}(B) \in \ker \mathcal{T}$ and smb is a homomorphism. We are going to show that this mapping is injective. Let $\mathrm{smb}\,(E^{-1}\mathcal{L}(A)E) = \mathrm{smb}\,\mathcal{T}(A) = 0$ for an operator $A \in \mathsf{TL}(X)$. Then $\mathcal{T}(A) \in \ker \mathcal{T}$; hence, $\mathcal{T}(A) = 0$. The first column in the table in Theorem 7 then shows that $\mathcal{L}(A) = \mathcal{L}(\mathcal{T}(A)) = 0$. ∎

It is not hard to see that the inverse of the mapping $A \mapsto \mathrm{smb}\,(E^{-1}AE)$ in Theorem 29 is explicitly given by

$$\mathrm{smb}\,\mathsf{TL}(X) = \mathcal{S}mb \to \mathrm{im}\,\mathcal{L}, \quad q \mapsto \mathcal{L}(\mathsf{T}(q)).$$

So we have two ways to think of the symbol of a Toeplitz-like operator: first, as an element of the quotient algebra $\mathsf{TL}(X)/\ker\mathcal{T}$; second as an operator in $\mathcal{L}(\mathsf{TL}(X))$ acting on X^2.

Corollary 30 *The mappings* (5) *are bounded.*

Proof. We shall demonstrate this for the mapping $\mathsf{T} : \mathcal{S}mb \to \mathrm{im}\,\mathcal{T},\ q \mapsto \mathsf{T}(q)$. Let $\mu : \mathcal{S}mb \to \mathrm{im}\,\mathcal{L}$ denote the inverse of the bounded isomorphism in Theorem 29 *(b)*. Then μ is bounded by a theorem by Banach; hence, the mapping

$$\mathcal{S}mb \to \mathrm{im}\,\mathcal{T}, \quad q \mapsto \mathsf{T}(q) = P\mu(q)|_X$$

is bounded. ∎

Remark 31 With the above notation and results, it becomes evident that Theorem 9 (b) can be viewed as a special case of Theorem 16, with $D : \operatorname{im}\mathcal{L} \to \mathsf{TL}(X),\ A \mapsto E^{-1}AE$ as the discretization mapping and $\mathcal{L}$ as its homomorphic left inverse. ■

4 Laurent-like operators

4.1 The axioms

In the previous sections, we started with two families $\mathcal{V}$ and $\mathcal{R}$ of operators to define first an algebra of Toeplitz-like operators and related abstract Toeplitz and Hankel operators, and then a group of shifts on a larger space, together with the related abstract Laurent operators. We will see now that one can also go the way around.

This approach works for Banach spaces X' which are a direct sum of two closed subspaces *of equal size*. Formally this means that there are operators $P,\ Q,\ J \in L(X')$ such that

$$P^2 = P,\ Q^2 = Q,\ P + Q = I \quad \text{and} \quad J^2 = I,\ JPJ = Q.$$

Then X' is equal to the direct sum $\operatorname{im} P \oplus \operatorname{im} Q$, the mapping $J|_{\operatorname{imP}} : \operatorname{im} P \to \operatorname{im} Q$ is a linear isomorphism and, with $X := \operatorname{im} P$, there is a linear isomorphism

$$X' \to X^2, \quad x \mapsto (Px,\ JQx) = (Px,\ PJx).$$

In what follows we simply assume that X' is already of the form X^2 for a certain Banach space X and that P, Q and J are given by

$$P : (x,\ y) \mapsto (x,\ 0), \quad Q : (x,\ y) \mapsto (0,\ y) \quad \text{and} \quad J : (x,\ y) \mapsto (y,\ x).$$

Again we write the elements of X^2 as column vectors and identify operators in $L(X^2)$ with 2×2 matrices with entries in $L(X)$.

Let $G \neq \{0\}$ and G_+ be as before, and let $(U_t)_{t\in G}$ be a bounded family of operators $U_t \in L(X^2)$ subject to the following axioms:

(L_1) The mapping $t \mapsto U_t$ is a group isomorphism on G. In particular, $U_sU_t = U_{s+t}$ for all $s,\ t \in G$.

(L_2) $U_0 = I$ and $JU_tJ = U_{-t}$ for every $t \in G$.

(L_3) $PU_sPU_tP = PU_{s+t}P$ for all $s,\ t \in G_+$.

(L_4) $U_{-t}PU_t \to I$ strongly as $t \to \infty$.

The first condition in axiom (L_2) ensures that all operators U_t are invertible in $L(X^2)$ and that $U_t^{-1} = U_{-t}$. By calling an invertible operator $U \in L(X^2)$ *J-unitary*, we can rephrase the second condition in axiom (L_2) as the J-unitarity of the U_t. It is easy to see that if U is J-unitary, then there are operators $V_\pm$, $R_\pm \in L(X)$ such that

$$U = \begin{pmatrix} V_+ & R_+ \\ R_- & V_- \end{pmatrix} \quad \text{and} \quad U^{-1} = \begin{pmatrix} V_- & R_- \\ R_+ & V_+ \end{pmatrix}.$$

In particular, there are operators V_t, $R_t \in L(X)$ such that

$$U_t = \begin{pmatrix} V_t & R_t \\ R_{-t} & V_{-t} \end{pmatrix} \quad \text{and} \quad U_t^{-1} = U_{-t} = \begin{pmatrix} V_{-t} & R_{-t} \\ R_t & V_t \end{pmatrix}$$

for all $t \in G_+$.

Lemma 32 *The so-defined operators $V_{\pm t}$ and $R_{\pm t}$ are subject to axioms (T_1) – (T_4).*

Proof. The group property of the U_t implies that

$$V_s V_t + R_s R_{-t} = V_{s+t}, \quad V_s R_t + R_s V_{-t} = R_{s+t} \quad \text{for } s,\, t \in G. \tag{13}$$

Since $V_s V_t = V_{s+t}$ by (L_3), the first identity in (13) implies that

$$R_s R_{-t} = 0 \quad \text{for } s,\, t \in G_+. \tag{14}$$

Further we conclude from (L_4) that

$$\begin{pmatrix} V_{-t} & R_{-t} \\ R_t & V_t \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_t & R_t \\ R_{-t} & V_{-t} \end{pmatrix} = \begin{pmatrix} V_{-t}V_t & V_{-t}R_t \\ R_{-t}V_t & R_t^2 \end{pmatrix} \to \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

as $t \to \infty$, whence $R_t^2 \to I$ as $t \to \infty$. Together with (13) this implies that

$$0 = R_s^2 R_{-t} \to R_{-t} \quad \text{as } s \to \infty.$$

Thus, $R_{-t} = 0$ for $t \in G_+$, and the matrix representations of U_t and U_{-t} are upper and lower triangular, respectively. With this information it is easy to check that the $(V_{-t})_{t \in G_+}$ own the semigroup property and that $V_{-t}V_t = I$ for $t \in G_+$ and $V_t V_{-t} \to 0$ as $t \to \infty$. Then $V_{-t}V_tV_{-t} = V_{-t} \to 0$ as $t \to \infty$, i.e., (T_4) holds. Another consequence of $V_tV_{-t} \to 0$ is that $V_tV_{-t} \neq I$ respective $P_t := I - V_tV_{-t} \neq 0$ for large t. It follows as in the proof of Lemma 23 that then $P_t \neq 0$ for all positive t, i.e. $V_tV_{-t} \neq I$ for all positive t.

It remains to check (T_3). Let $t \in G_+$. The identity $V_tV_{-t} + R_t^2 = I$ is nothing but the south-east corner of 2×2-matrix identity $U_{-t}U_t = I$. Further, $R_tV_t = 0$ by the south-west corner of that identity. Hence, $R_tQ_t = 0$ and $R_tP_t = P_tR_t = R_t$. Now the second identity in (T_3) follows easily by multiplying the second identity in (13) from the right-hand side by P_t. ∎

Note that, conversely, the operators defined by (8) satisfy the axioms (L_1) – (L_4).

4.2 Laurent-like operators

Let X^2 and the operators U_t be as in Section 4.1. We let $\mathsf{LL}(X^2)$ stand for the set of all operators $A \in L(X^2)$ for which the two strong limits

$$\mathcal{L}_\pm(A) := \text{s-lim}_{t\to\pm\infty} U_{-t}AU_t$$

exist. We call the elements of $\mathsf{LL}(X^2)$ *Laurent-like operators.*

Theorem 33 *(a)* $\mathsf{LL}(X^2)$ *is a closed subalgebra of* $L(X^2)$.
(b) An operator $A \in L(X^2)$ *is in* $\mathsf{LL}(X^2)$ *if and only if*

$$A = \begin{pmatrix} B & C \\ D & E \end{pmatrix} \quad \text{with} \quad B,\, C,\, D,\, E \in \mathsf{TL}(X) \quad \text{and} \quad C,\, D \in \ker \mathcal{T}. \tag{15}$$

(c) If A is of the form (15)*, then*

$$\mathcal{L}_+(A) = \begin{pmatrix} \mathcal{T}(B) & \mathcal{H}(B) \\ \widetilde{\mathcal{H}}(B) & \widetilde{\mathcal{T}}(B) \end{pmatrix} \quad \text{and} \quad \mathcal{L}_-(A) = \begin{pmatrix} \widetilde{\mathcal{T}}(E) & \widetilde{\mathcal{H}}(E) \\ \mathcal{H}(E) & \mathcal{T}(E) \end{pmatrix}. \tag{16}$$

Proof. Assertion (a) is standard. Let A be of the form (15). Then

$$U_{-t}AU_t = \begin{pmatrix} V_{-t}BV_t & V_{-t}BR_t + V_{-t}CV_{-t} \\ R_tBV_t + V_tDV_t & R_tBR_t + V_tDR_t + R_tCV_{-t} + V_tEV_{-t} \end{pmatrix}.$$

The operators $V_{-t}BV_t$, $V_{-t}BR_t$, R_tBV_t and R_tBR_t converge strongly to $\mathcal{T}(B)$, $\mathcal{H}(B)$, $\widetilde{\mathcal{H}}(B)$ and $\widetilde{\mathcal{T}}(B)$ as $t \to \infty$, respectively, because $B \in \mathsf{TL}(X)$. Since $V_{-t} \to 0$ strongly, the operators $V_{-t}CV_{-t}$ and V_tEV_{-t} converge strongly to 0. Finally, the operators V_tDV_t and V_tDR_t converge strongly to 0 because $D \in \ker \mathcal{T}$ and by Lemma 1.11 in [18]. This proves the first assertion in (c); the second one follows analogously.

It remains to verify the "only if" implication in (b). From (L_4) and the J-unitarity of the U_t we conclude that P and Q belong to $\mathsf{LL}(X^2)$. Thus, if

$$A = \begin{pmatrix} B & C \\ D & E \end{pmatrix} \in \mathsf{LL}(X^2),$$

then

$$\begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ D & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & E \end{pmatrix} \in \mathsf{LL}(X^2).$$

From

$$U_{-t} \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} U_t = \begin{pmatrix} V_{-t}BV_t & V_{-t}BR_t \\ R_tBV_t & R_tBR_t \end{pmatrix}$$

we conclude that $B \in \mathsf{TL}(X)$. Further, the equality

$$U_{-t} \begin{pmatrix} 0 & 0 \\ D & 0 \end{pmatrix} U_t = \begin{pmatrix} 0 & 0 \\ V_tDV_t & V_tDR_t \end{pmatrix}$$

implies that the operators $V_t D V_t$ and $V_t D R_t$ converge strongly. Since $V_{-t} \to 0$ strongly, we conclude that

$$DV_t = V_{-t}V_tDV_t \to 0 \quad \text{and} \quad DR_t = V_{-t}V_tDR_t \to 0 \quad \text{strongly as } t \to \infty.$$

But then

$$V_{-t}DV_t \to 0, \quad R_tDV_t \to 0, \quad V_{-t}DR_t \to 0, \quad R_tDR_t \to 0 \quad \text{strongly as } t \to \infty.$$

Hence, $D \in \mathsf{TL}(X)$, and each of the four strong limits is zero. In particular, $D \in \ker \mathcal{T}$. Working with $\mathcal{L}_-$ instead of $\mathcal{L}_+$, we obtain that C, $E \in \mathsf{TL}(X)$ and $C \in \ker \mathcal{T}$. ■

5 Examples of Toeplitz-like operators

Here we present a few concrete Banach spaces and examine the related algebras of Toeplitz-like operators. A main objective is to identify the abstract Toeplitz and Hankel operators $\mathsf{T}(a)$ and $\mathsf{H}(a)$ with classical (concrete) Toeplitz and Hankel operators acting on these spaces. For a general acquaintance with (concrete) Toeplitz and Hankel operators see, e.g., [4, 8, 14, 16].

5.1 Toeplitz-like operators on H^p

Let H^p, $1 < p < \infty$, be the Hardy space introduced in Section 1, and let $G = \mathbb{Z}$. Let $V_0 := I$ and, for $n \in \mathbb{Z}_+ \setminus \{0\}$, define operators $V_{\pm n}$ on H^p by

$$V_n f := \chi_n f \quad \text{and} \quad V_{-n} f := \chi_{-n} \left(f - \sum_{i=0}^{n-1} \hat{f}_i \chi_i \right),$$

respectively. Clearly, $V_{-n}V_n = I$ and $V_{-n} \to 0$ strongly as $n \to \infty$. Note also that

$$\ker V_{-n} = \left\{ f \in H^p : f = \sum_{i=0}^{n-1} \hat{f}_i \chi_i \right\}.$$

Finally, we define operators R_n on H^p by

$$R_n f := \sum_{i=0}^{n-1} \hat{f}_{n-1-i} \chi_i$$

if $n \in \mathbb{Z}_+ \setminus \{0\}$ and $R_n = 0$ else. Then all axioms of Section 2.1 are satisfied, and the corresponding algebra $\mathsf{TL}(H^p)$ is well defined. Moreover, $V_n \to 0$ and $R_n \to 0$ weakly. We let U_n stand for the related operators on $(H_p)^2$, defined as in Section 3.5.

As already mentioned, the space $H^p \subset L^p(\mathbb{T})$ is equal to $\operatorname{im} P$, with P the classical Riesz projection. Set $Q := I - P$. Thus, $L^p(\mathbb{T}) = \operatorname{im} P \oplus \operatorname{im} Q$, whereas $(H^p)^2 = \operatorname{im} P \oplus \operatorname{im} P$, which we may write as

$$L^p(\mathbb{T}) = \begin{pmatrix} \operatorname{im} P \\ \operatorname{im} Q \end{pmatrix} \quad \text{and} \quad (H^p)^2 = \begin{pmatrix} \operatorname{im} P \\ \operatorname{im} P \end{pmatrix}.$$

With respect to this identification, we may further identify operators from $L^p(\mathbb{T})$ to $(H^p)^2$ with 2×2-matrices. It is easy to see that the operators

$$\eta := \begin{pmatrix} P & 0 \\ 0 & JQ \end{pmatrix} : \begin{pmatrix} \operatorname{im} P \\ \operatorname{im} Q \end{pmatrix} \to \begin{pmatrix} \operatorname{im} P \\ \operatorname{im} P \end{pmatrix}$$

and

$$\eta^{-1} := \begin{pmatrix} P & 0 \\ 0 & JP \end{pmatrix} : \begin{pmatrix} \operatorname{im} P \\ \operatorname{im} P \end{pmatrix} \to \begin{pmatrix} \operatorname{im} P \\ \operatorname{im} Q \end{pmatrix}$$

are inverse to each other and that $\eta^{-1} U_n \eta = \chi_n I$ for all $n \in \mathbb{Z}$.

Now let $A \in \mathsf{TL}(H^p)$. From $U_{-n} \mathcal{L}(A) U_n = \mathcal{L}(A)$ for $n \in \mathbb{Z}_+$ we conclude that

$$\chi_{-n} (\eta^{-1} \mathcal{L}(A) \eta) \chi_n I = \eta^{-1} \mathcal{L}(A) \eta.$$

Hence, $\eta^{-1} \mathcal{L}(A) \eta$ is the operator of multiplication by some function $q \in L^\infty(\mathbb{T})$ (see [5]), and

$$\eta^{-1} \mathcal{L}(A) \eta = qI = \begin{pmatrix} PqP & PqQ \\ QqP & QqQ \end{pmatrix},$$

which implies

$$\mathcal{L}(A) = \begin{pmatrix} T(q) & H(q) \\ H(\tilde{q}) & T(\tilde{q}) \end{pmatrix}.$$

In particular, $\mathcal{T}(A)$, $\widetilde{\mathcal{T}}(A)$, $\mathcal{H}(A)$ and $\widetilde{\mathcal{H}}(A)$ coincide with the familiar (concrete) Toeplitz and Hankel operators $T(q)$, $T(\tilde{q})$, $H(q)$ and $H(\tilde{q})$ on H^p as defined in Section 1.

As a consequence, we observe that the corresponding symbol algebra $\mathcal{S}mb$ is a subalgebra of $L^\infty(\mathbb{T})$. In fact, $\mathcal{S}mb = L^\infty(\mathbb{T})$, which is an immediate consequence of the identities

$$V_{-n} T(a) V_n = T(a), \qquad R_n T(a) R_n = P_n T(\tilde{a}), \tag{17}$$

$$V_{-n} T(a) R_n = H(a) P_n, \qquad R_n T(a) V_n = P_n H(\tilde{a}) \tag{18}$$

for all $a \in L^\infty(\mathbb{T})$ and $n \in \mathbb{Z}_+$. Let us check the last identity of (18), for example. By the first relation in (1) we have

$$I = T(\chi_n \chi_{-n}) = T(\chi_n) T(\chi_{-n}) + H(\chi_n) H(\chi_n).$$

Hence, $H(\chi_n)^2 = P_n$. Further,

$$R_n T(a) V_n = R_n T(a \chi_n) = R_n (T(\chi_n) T(a) - H(\chi_n) H(\tilde{a})).$$

Since $R_n = H(\chi_n)$ and $R_n T(\chi_n) = 0$, we conclude that $R_n T(a) V_n = P_n H(\tilde{a})$, as desired. Note that one could also start with (17), (18) in order to derive that $\eta^{-1}\mathcal{L}(T(a))\eta = aI$.

Let $\mathcal{C}$ be a closed subalgebra of $L^\infty(\mathbb{T})$. Then, in the present context, Corollary 19 specifies to

$$\operatorname{alg} TH(\mathcal{C}) = T(\mathcal{C}) \oplus \ker(\mathcal{T}|_{\operatorname{alg} TH(\mathcal{C})}).$$

This result was recently proved by Didas [7, Theorem 2.4] for $p = 2$ and under the assumption that $\mathcal{C}$ is an inner subalgebra of $L^\infty(\mathbb{T})$ in the sense of [15]. For example, $C(\mathbb{T})$ is inner, and so is every closed subalgebra of $L^\infty(\mathbb{T})$ which strictly contains H^∞. Moreover, Didas showed that $\ker(\mathcal{T}|_{\operatorname{alg} TH(\mathcal{C})})$ equals the commutator ideal of $\operatorname{alg} TH(\mathcal{C})$ under this additional assumption.

At the end of Section 2.2 we remarked that $\ker \mathcal{H}$ and $\ker \widetilde{\mathcal{H}}$ are subalgebras of $\mathsf{TL}(X)$. In the setting of this section, we can describe these algebras more precisely as

$$\ker \mathcal{H} = \{T(a) : a \in H^\infty\} \oplus \ker \mathcal{T}, \quad \ker \widetilde{\mathcal{H}} = \{T(a) : a \in \overline{H^\infty}\} \oplus \ker \mathcal{T}.$$

Remark 34 The construction of the algebra $\mathsf{TL}(X)$ works as well on weighted Hardy spaces $H^p(w)$ and the above results remain valid in this setting provided that the weight w is symmetric (i.e. $w(t) = w(1/t)$) and that the Riesz projection P is bounded on $L^p(w)$ (which holds if w is a Muckenhoupt weight, for example if w is a power weight; see [3], Chapter 2). In this setting, the operators V_n and R_n can be defined as above; they are uniformly bounded and all requirements made in Section 2.1 are satisfied.

Remark 35 If $1 < p < \infty$ and $\mathcal{S}$ is a closed subalgebra of the symbol algebra $L^\infty(\mathbb{T})$ which contains $C(\mathbb{T})$, then the Toeplitz algebra $\operatorname{alg} \mathsf{T}(\mathcal{S}) \subset L(H^p)$ contains all compact operators. An analogous result holds in the following two examples if $\mathcal{S}$ is a closed subalgebra of corresponding multiplier algebras which contains all functions in the Wiener algebra over $\mathbb{T}$ and $\mathbb{R}$, respectively.

5.2 Toeplitz-like operators on l^p

Here we are going to employ the alternative approach of Section 3.5 to define Toeplitz-like operators on the classical sequence spaces $l^p(\mathbb{Z}_+)$, $1 \le p < \infty$. Consider the reflection operator $J : (x_n)_{n\in\mathbb{Z}} \mapsto (y_n)_{n\in\mathbb{Z}}$ where $y_n = x_{-n-1}$ and the discrete Riesz projection $J : (x_n)_{n\in\mathbb{Z}} \mapsto (y_n)_{n\in\mathbb{Z}}$ where $y_n = x_n$ if $n \ge 0$ and $y_n = 0$ else. Clearly, $J^2 = I$ and $JPJ = I - P =: Q$.

Let $a \in L^\infty(\mathbb{T})$. On $l_0(\mathbb{Z})$, the linear space of the finitely supported sequences, we consider the Laurent operator $L(a)$ defined by

$$(L(a)x)_k := \sum_{m\in\mathbb{Z}} \hat{a}_{k-m} x_m.$$

Note that only finitely many items in this sum do not vanish. The function a is called a *multiplier* on $l^p(\mathbb{Z})$ if

$$\|L(a)\| := \sup\{\|L(a)x\|_p : x \in l_0(\mathbb{Z}),\ \|x\|_p\} < \infty.$$

If a is a multiplier, then $L(a)$ extends to a bounded linear operator on $l^p(\mathbb{Z})$ which we denote by $L(a)$ again. The set M^p of all multipliers on $l^p(\mathbb{Z})$ forms a Banach algebra under the norm $\|a\|_{M^p} := \|L(a)\|_{L(l^p(\mathbb{Z}))}$ (see [4, Chapter 2]). In particular, every function $a \in L^\infty(\mathbb{T})$ with bounded total variation belongs to M^p, and there is a constant c_p independent of a such that Stechkin's inequality

$$\|a\|_{M^p} \le c_p(\|a\|_\infty + \mathrm{Var}\,(a))$$

holds. Note also that $JL(a)J = L(\tilde{a})$ for every $a \in M^p$.

Write l^p for $l^p(\mathbb{Z}_+) = \mathrm{im}\, P$ and let $a \in M^p$. The operators

$$T(a) : l^p \to l^p,\ f \mapsto PL(a)f, \quad H(a) : l^p \to l^p,\ f \mapsto PL(a)QJf$$

are called the (concrete) *Toeplitz* and *Hankel operator* with generating function a, respectively. These operators are bounded if $a \in M^p$. It is also easy to see that if $T(a)f = g$ and $H(a)f = h$ then

$$g_j = \sum_{k=0}^{\infty} \hat{a}_{j-k} f_k, \quad h_j = \sum_{k=0}^{\infty} \hat{a}_{j+k+1} f_k.$$

Let $\{e_j\}_{j\in\mathbb{Z}_+}$ stand for the standard basis on l^p, i.e. the jth entry of e_j is 1 whereas all other entries are 0. The l^p-version of a Theorem by Brown and Halmos (Theorem 2.7 (b) in [4]) tells us that if $A \in L(l^p)$ and if there is a sequence $(a_n)_{n\in\mathbb{Z}}$ in $\mathbb{C}$ such that

$$\langle Ae_j,\, e_k\rangle = a_{j-k} \quad \text{for all } j,\, k \in \mathbb{Z}_+,$$

then there exists a multiplier $a \in M^p$ such that $A = T(a)$ and a_n is the nth Fourier coefficient of a. Here, as usual, $\langle x,\, y\rangle := \sum x_n y_n$ for $x \in l^p$ and $y \in l^q$ with $p^{-1} + q^{-1} = 1$.

Next we are going to define the related algebra $\mathsf{TL}(l^p)$ of Toeplitz-like operators. Set $G = \mathbb{Z}$ and define

$$V_n := T(\chi_n) \quad \text{and} \quad R_n := H(\chi_n) \quad \text{for } n \in \mathbb{Z}. \tag{19}$$

One easily checks that these operators satisfy axioms (T_1) – (T_4); hence, the algebra $\mathsf{TL}(l^p)$ with respect to these families is well defined. Moreover, if $1 < p < \infty$, then $V_n \to 0$ and $R_n \to 0$ weakly as $n \to \infty$. Note that this claim fails for $p = 1$. Indeed, consider the bounded linear functional

$$g : l^1(\mathbb{Z}_+) \to \mathbb{C}, \quad x = (x_i)_{i\in\mathbb{Z}_+} \mapsto \sum_{i\in\mathbb{Z}_+} x_i.$$

Then $g(V_n x) = g(x)$ for all $x \in l^1(\mathbb{Z}_+)$ and $n \in \mathbb{Z}_+$; hence, the V_n cannot converge weakly to zero. The argument for the R_s is similar.

If $A \in \mathsf{TL}(l^p)$ and $\mathcal{T}(A) = A$, then $V_1 A V_1 = A$. Thus, A satisfies the hypotheses of the Brown/Halmos theorem, and there is a multiplier $a \in M^p$ such that $A = T(a)$. Thus, every abstract Toeplitz (or Hankel) operator on l^p is a concrete Toeplitz (or Hankel) operator as defined above. The l^p-version of Theorem 9 implies

$$\mathsf{TL}(l^p)/\ker \mathcal{T} \cong M^p.$$

Remark 36 An equivalent treatment of Toeplitz-like operators on *weighted* l^p-spaces fails: the operators (19) are not longer uniformly bounded then. This is in strong contrast to the H^p-setting in the previous section. ■

Our final goal is to provide some of the counter examples promised above. In particular, we are going to show that

(a) $\mathsf{TL}(l^2)$ is not a C^*-algebra.

(b) $\ker \mathcal{T} \subset \mathsf{TL}(l^2)$ is not a two-sided ideal of $L(l^2)$.

(c) $\operatorname{alg} TH(L^\infty)$ is a proper subalgebra of $\mathsf{TL}(l^2)$.

(d) $\operatorname{alg} T(L^\infty)$ is a proper subalgebra of $\operatorname{alg} TH(L^\infty)$.

For (a), assume that $\mathsf{TL}(l^2)$ is a C^*-algebra. Then $\ker \mathcal{T}$ is a closed *symmetric* ideal of $\mathsf{TL}(l^2)$ and, hence, a closed *symmetric* left-sided ideal of $L(l^2)$. So we have BC, $B^*C^* \in \ker \mathcal{T}$ for every $B \in L(l^2)$ and $C \in \ker \mathcal{T}$ by the left ideal property, whence $(B^*C^*)^* = CB \in \ker \mathcal{T}$ by the symmetry. Thus, $\ker \mathcal{T}$ turns out to be a closed *two-sided* ideal of $L(l^2)$. Moreover, $\ker \mathcal{T}$ contains non-compact operators (for example, the Hankel operator which generates the Hilbert matrix), but not every bounded linear operator on l^2 is in $\ker \mathcal{T}$ (for example, $I \notin \ker \mathcal{T}$). This contradicts the well known fact that the only non-trivial closed two-sided ideal of $L(l^2)$ is the ideal of the compact operators.

The same arguments show assertion (b), and (c) follows because $\operatorname{alg} TH(L^\infty)$ is a symmetric closed subalgebra of $\mathsf{TL}(l^2)$, whereas $\mathsf{TL}(l^2)$ fails to be symmetric by (a). Finally, (d) comes from the existence of Hankel operators which do not belong to $\operatorname{alg} T(L^\infty)$ as first observed by S. Axler and proved in Power [16].

The following explicit example of an operator $A \in \mathsf{TL}(l^2)$ for which $A^* \notin \mathsf{TL}(l^2)$ is due to our former colleague Hans-Jürgen Fischer.

Example 37 Let the operator A be given by the matrix representation $A = (a_{ij})_{i,j=0}^\infty$ where

$$a_{ij} = \begin{cases} \frac{1}{\sqrt{2^i}} & \text{if } 0 \leq j \leq 2^i - 1, \\ 0 & \text{if } j \geq 2^i. \end{cases}$$

We show that A is a bounded operator on l^2. Let $x = (x_i)_{i\geq 0} \in l^2$ and $\|x\| = 1$. Then

$$\|Ax\|^2 = \sum_{n=0}^{\infty} \frac{1}{2^n} \left| \sum_{k=0}^{2^n-1} x_k \right|^2 .$$

For $k \in \mathbb{Z}^+$, set $\lambda_k := (k+1)^{-\frac{1}{4}}$. Then, by the Cauchy-Schwartz inequality,

$$\left| \sum_{k=0}^{2^n-1} x_k \right|^2 = \left| \sum_{k=0}^{2^n-1} \lambda_k \lambda_k^{-1} x_k \right|^2 \leq \left(\sum_{k=0}^{2^n-1} \lambda_k^2 \right) \left(\sum_{k=0}^{2^n-1} \left| \frac{x_k}{\lambda_k} \right|^2 \right).$$

The estimate

$$\sum_{k=0}^{2^n-1} \lambda_k^2 = \sum_{k=0}^{2^n-1} (k+1)^{-\frac{1}{2}} \leq 2\sqrt{2^n} - 1$$

then gives

$$\begin{aligned} \|Ax\|^2 &\leq \sum_{n=0}^{\infty} \frac{2\sqrt{2^n}-1}{2^n} \left(\sum_{k=0}^{2^n-1} \left| \frac{x_k}{\lambda_k} \right|^2 \right) \\ &\leq 2 \sum_{n=0}^{\infty} \sum_{k=0}^{2^n-1} \frac{1}{\sqrt{2^n}} \left| \frac{x_k}{\lambda_k} \right|^2 \\ &= 2 \sum_{k=0}^{\infty} \sum_{n \geq \log_2(k+1)} \frac{1}{\sqrt{2^n}} \left| \frac{x_k}{\lambda_k} \right|^2 . \end{aligned}$$

Using

$$\sum_{n \geq \log_2(k+1)} \frac{1}{\sqrt{2^n}} = \frac{1}{\sqrt{k+1}} \frac{1}{1-1/\sqrt{2}} = \frac{1}{\sqrt{k+1}} \sqrt{2}(\sqrt{2}+1),$$

we arrive at

$$\|Ax\|^2 \leq 2\sqrt{2}(\sqrt{2}+1) \sum_{k=0}^{\infty} \frac{1}{\sqrt{k+1}} \left| \frac{x_k}{\lambda_k} \right|^2 = 2\sqrt{2}(\sqrt{2}+1) \sum_{k=0}^{\infty} |x_k|^2,$$

whence the boundedness of A and the estimate $\|A\|^2 \leq 2\sqrt{2}(\sqrt{2}+1)$. From

$$\|AV_n e_k\|^2 = \|Ae_{n+k}\|^2 = 2^{-\log_2(n+k)} \quad \text{for } k,\, n \geq 0$$

and

$$\|AR_n e_k\|^2 = \|Ae_{n-k}\|^2 = 2^{-\log_2(n-k)} \quad \text{for } n > k \geq 0$$

we further conclude that $AV_n \to 0$ and $AR_n \to 0$ strongly as $n \to \infty$. Hence, $A \in \mathsf{TL}(l^2)$ (and $A \in \ker \mathcal{T}$). Suppose that $A^* \in \mathsf{TL}(l^2)$. Then $(V_{-n} A^* V_n)$ converges strongly to some operator $\mathcal{T}(A^*)$ by definition. On the other hand, $(V_{-n} A^* V_n) = (V_{-n} A V_n)*\to 0$ weakly because $V_{-n} A V_n \to 0$ strongly. Hence, $\mathcal{T}(A^*) = 0$. Then, by Lemma 1.11 in [18], $A^* V_n \to 0$ strongly, which is impossible since $\|A^* V_n e_0\| = 1$ for all $n \in \mathbb{Z}^+$. ■

Remark 38 It is an open question if $\operatorname{alg} TH(L^\infty)$ is a proper subalgebra of

$$\mathsf{TL}^*(l^2) := \{A \in \mathsf{TL}(l^2) : A^* \in \mathsf{TL}(l^2)\}.$$

5.3 Toeplitz-like operators on $L^p(\mathbb{R}_+)$

Throughout this section, we let L^p, L^p_+ and L^p_- stand for the classical Lebesgue spaces on $\mathbb{R}$, $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}_- = \mathbb{R} \setminus \mathbb{R}_+$, respectively. Every function $a \in L^\infty$ induces a bounded operator $f \mapsto af$ on L^p which we denote by $m(a)$. In particular, if χ_+ and χ_- stand for the characteristic functions of $\mathbb{R}_+$ and $\mathbb{R}_-$, then we let P and Q denote the operators $m(\chi_+)$ and $m(\chi_-)$, respectively. The ranges of P and Q can be identified with L^p_+ and L^p_-. We will also need the flip operator J on L^p which is defined by $(Jf)(t) = f(-t)$. Clearly, $J^2 = I$ and $Jm(a)J = m(\tilde{a})$ where $\tilde{a} = a(-t)$ for $a \in L^\infty$.

We write the Fourier transform on L^1 in the form

$$(Ff)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{ixt}\, dt, \quad x \in \mathbb{R},$$

and use the same letter F to denote the continuous extension of F to a unitary operator on L^2. Note that $F^{-1} = JF$.

Let $a \in L^\infty$. Then the operator $M(a) := F^{-1} m(a) F : L^2 \to L^2$ is bounded and has norm $\|a\|_\infty$. For $1 \le p < \infty$, we let $M^p(\mathbb{R})$ stand for the collection of all functions $a \in L^\infty$ owning the following property: Whenever $f \in L^2 \cap L^p$, then $M(a) \in L^p$, and there is a constant c_p independent of f such that $\|M(a)f\|_p \le c_p \|f\|_p$ for all $f \in L^2 \cap L^p$. If $a \in M^p(\mathbb{R})$, then $M(a) : L^2 \cap L^p \to L^p$ extends to a bounded operator on L^p, called the operator of convolution by a. We denote this operator by $M(a)$ again. The elements of $M^p(\mathbb{R})$ are also called L^p-multipliers.

An operator A on L^p is called *translation invariant* if $U_{-s} A U_s = A$ for all $s \in \mathbb{R}$ where $(U_s f)(t) := f(t - s)$. If $a \in M^p(\mathbb{R})$, then the convolution operator $M(a)$ is translation invariant. A theorem of Hörmander [13] (see also 9.2 in [4]) states that the converse is also true: every translation invariant operator on L^p is an operator of convolution by a certain L^p-multiplier.

If $a \in L^\infty$ is of finite total variation $V(a)$, then $a \in M^p(\mathbb{R})$, and the Stechkin inequality

$$\|M(a)\|_L(L^p) \le C_p(\|a\|_\infty + V(a))$$

holds. Here, C_p is a constant independent of a which can be chosen concretely as $C_p = \|M(\chi_+ - \chi_-)\|_L(L^p)$. Note also that $JM(a)J = M(\tilde{a})$. Further basic properties of $M^p(\mathbb{R})$ can be found in [4, Chapter 9], for example.

Let $1 \le p < \infty$. With every multiplier $a \in M^p(\mathbb{R})$, we associate the *Wiener-Hopf integral operator* $W(a)$ and the *Hankel integral operator* $H_{\mathbb{R}}(a)$ by

$$W(a) := WM(a)|_{L^p_+} : L^p_+ \to L^p_+ \quad \text{and} \quad H_{\mathbb{R}}(a) := PM(a)QJ|_{L^p_+} : L^p_+ \to L^p_+,$$

respectively. These operators are bounded on L^p_+, and the following analogues of (1) hold for all L^p-multipliers a, b:

$$W(ab) = W(a)W(b) + H_{\mathbb{R}}(a)H_{\mathbb{R}}(\tilde{b}), \quad H_{\mathbb{R}}(ab) = W(a)H_{\mathbb{R}}(b) + H_{\mathbb{R}}(a)W(\tilde{b}).$$

For $s \in \mathbb{R}$, define $\omega_s(x) := e^{isx}$. Then $\omega_s \in M^p(\mathbb{R})$ for every $p \in [1, \infty)$ and $M(\omega_s)$ is nothing but the translation operator U_s. Now set $V_s := W(\omega_s)$ for $s \in G$, $R_s := H_{\mathbb{R}}(\omega_s)$ for $s \in \mathbb{R}_+$ and $R_s = 0$ for $s \in \mathbb{R}_-$. Then the axioms in Section 2.1 are satisfied, thus, the algebra $\mathsf{TL}(L^p_+)$ is well defined. Moreover, the operators V_s and R_s tend weakly to 0 as $s \to \infty$ if $1 < p < \infty$. For $p = 1$, neither the V_s nor the R_s tend weakly to zero which follows by arguments similar to those for $X = l^1$.

As before, one can easily prove that for $s \geq 0$ and $a \in M^p(\mathbb{R})$,

$$V_{-s}W(a)V_s = W(a), \quad R_sW(a)R_s = P_sW(\tilde{a})P_s,$$

$$V_{-s}W(a)R_s = H_{\mathbb{R}}(a)P_s, \quad R_sW(a)V_s = P_sH_{\mathbb{R}}(\tilde{a}).$$

What results is that every Wiener-Hopf operator $W(a)$ belongs to $\mathsf{TL}(L^p_+)$ and that

$$\mathcal{T}(W(a)) = W(a), \quad \widetilde{\mathcal{T}}(W(a)) = W(\tilde{a}),$$
$$\mathcal{H}(W(a)) = H_{\mathbb{R}}(a), \quad \widetilde{\mathcal{H}}(W(a)) = H_{\mathbb{R}}(\tilde{a}).$$

Conversely, we are going to show that if $A \in \mathsf{TL}(L^p_+)$ and $\mathcal{T}(A) = A$, then there exists a multiplier $a \in M^p(\mathbb{R})$ such that $A = W(a)$. We proceed as in Section 5.1. Slightly abusing the notation, we let U_s stand for the related operators on $(L^p_+)^2$, defined as in Section 3.5. As already mentioned, the space $L^p_+ \subset L^p$ is equal to $\operatorname{im} P$. Thus, $L^p = \operatorname{im} P \oplus \operatorname{im} Q$, whereas $(L^p_+)^2 = \operatorname{im} P \oplus \operatorname{im} P$, which we may write as

$$L^p = \begin{pmatrix} \operatorname{im} P \\ \operatorname{im} Q \end{pmatrix} \quad \text{and} \quad (L^p_+)^2 = \begin{pmatrix} \operatorname{im} P \\ \operatorname{im} P \end{pmatrix}.$$

With respect to this identification, we may further identify operators from L^p to $(L^p_+)^2$ with 2×2-matrices. It is easy to see that the operators

$$\eta := \begin{pmatrix} P & 0 \\ 0 & JQ \end{pmatrix} : \begin{pmatrix} \operatorname{im} P \\ \operatorname{im} Q \end{pmatrix} \to \begin{pmatrix} \operatorname{im} P \\ \operatorname{im} P \end{pmatrix}$$

and

$$\eta^{-1} := \begin{pmatrix} P & 0 \\ 0 & JP \end{pmatrix} : \begin{pmatrix} \operatorname{im} P \\ \operatorname{im} P \end{pmatrix} \to \begin{pmatrix} \operatorname{im} P \\ \operatorname{im} Q \end{pmatrix}$$

are inverse to each other and that $\eta^{-1}U_s\eta = M(\omega_s)$ for all $s \in \mathbb{R}$.

Now let $A \in \mathsf{TL}(L^p_+)$. From $U_{-s}\mathcal{L}(A)U_s = \mathcal{L}(A)$ for $s \in \mathbb{R}_+$ we conclude that

$$M(\omega_{-s})(\eta^{-1}\mathcal{L}(A)\eta)M(\omega_s) = \eta^{-1}\mathcal{L}(A)\eta.$$

This equality implies that $\eta^{-1}\mathcal{L}(A)\eta$ is translation invariant. By Hörmander's theorem, there is a multiplier $a \in M^p(\mathbb{R})$ such that

$$\eta^{-1}\mathcal{L}(A)\eta = M(a) = \begin{pmatrix} PM(a)P & PM(a)Q \\ QM(a)P & QM(a)Q \end{pmatrix},$$

which implies

$$\mathcal{L}(A) = \begin{pmatrix} W(a) & H_{\mathbb{R}}(a) \\ H_{\mathbb{R}}(\tilde{a}) & W(\tilde{a}) \end{pmatrix}.$$

In particular, $\mathcal{T}(A)$, $\widetilde{\mathcal{T}}(A)$, $\mathcal{H}(A)$ and $\widetilde{\mathcal{H}}(A)$ coincide with the familiar (concrete) Wiener-Hopf and Hankel operators $W(a)$, $W(\tilde{q})$, $H_{\mathbb{R}}(a)$ and $H_{\mathbb{R}}(\tilde{a})$ as defined in the beginning of this section.

We thus arrived at the following theorem which can be viewed as the analogue of the Brown/Halmos theorem in the L^p_+-setting:

Theorem 39 *(a) If $a \in M^p(\mathbb{R})$, then $W(a)$ belongs to $\mathsf{TL}(L^p_+)$.*
(b) If $A \in \mathsf{TL}(L^p_+)$ and $A = \mathcal{T}(A)$, then there exists a multiplier $a \in M^p(\mathbb{R})$ such that $A = W(a)$.

Corollary 40 $\mathsf{TL}(L^p_+)/\ker \mathcal{T} \cong M^p(\mathbb{R})$.

In particular we conclude that $M^p(\mathbb{R})$ actually forms a Banach algebra.

6 Derivations

Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras. A *derivation* on $\mathcal{A}$ is a (not necessarily bounded) linear operator $D : \mathcal{A} \to \mathcal{B}$ such that $D(ab) = D(a)b + aD(b)$ for all $a, b \in \mathcal{A}$. Every element $c \in \mathcal{A}$ induces a derivation $a \mapsto ca - ac$ on $\mathcal{A}$.

Chernoff [6] showed that if X is an infinite-dimensional Banach space and $\mathcal{A}$ a closed subalgebra of $L(X)$ which contains the compact operators, then every derivation $D : \mathcal{A} \to L(X)$ is bounded, and there is an operator $C \in L(X)$ such that $D(A) = CA - AC$ for all $A \in \mathcal{A}$.

Theorem 41 *Let $\mathcal{A}$ be a closed subalgebra of $\mathsf{LT}(X)$ and suppose that the symbol algebra* $\operatorname{smb}\mathcal{A}$ *is commutative and semisimple. Then every bounded derivation $D : \mathcal{A} \to \mathcal{A}$ maps $\mathcal{A}$ into* $\ker \mathcal{T}$.

Proof. Let π denote the canonical homomorphism from $\mathcal{A}$ onto $\mathcal{A}/\mathrm{com}\,\mathcal{A} =: \mathcal{A}^\pi$. From

$$\begin{aligned} D(AB - BA) &= (D(A)B + AD(B)) - (D(B)A + BD(A)) \\ &= (D(A)B - BD(A)) + (AD(B) - D(B)A) \end{aligned}$$

we conclude that D maps $\operatorname{com}\mathcal{A}$ into $\operatorname{com}\mathcal{A}$. Hence, the quotient mapping

$$D^\pi : \mathcal{A}^\pi \to \mathcal{A}^\pi, \; \pi(A) \mapsto \pi(D(A))$$

is a well defined bounded derivation on $\mathcal{A}^\pi$. Since $\mathcal{A}/\operatorname{com}\mathcal{A}$ is commutative, D^π maps the algebra $\mathcal{A}^\pi$ into its Jacobson radical $\operatorname{rad}\mathcal{A}^\pi$ by the Singer-Wermer theorem (see [20]).

Since $\operatorname{smb}\mathcal{A}$ is commutative, the commutator ideal $\operatorname{com}\mathcal{A}$ is contained in $\mathcal{A}\cap\ker\mathcal{T}$, as we observed at the end of Section 3.2. Hence, π maps $\mathcal{A}\cap\ker\mathcal{T}$ onto a closed ideal $(\mathcal{A}\cap\ker\mathcal{T})^\pi$ of $\mathcal{A}^\pi$. It is not hard to see that the quotient algebra $\mathcal{A}^\pi/(\mathcal{A}\cap\ker\mathcal{T})^\pi$ is isomorphic to the algebra $\operatorname{smb}\mathcal{A}$, which is semisimple by assumption. Hence, $\operatorname{rad}\mathcal{A}^\pi \subseteq (\mathcal{A}\cap\ker\mathcal{T})^\pi$. Thus, D^π maps $\mathcal{A}^\pi$ into $(\mathcal{A}\cap\ker\mathcal{T})^\pi$, which implies that D maps $\mathcal{A}$ into $\mathcal{A}\cap\ker\mathcal{T}$. ∎

Combining this result with Chernoff's theorem, we arrive at the following.

Corollary 42 *Let $\mathcal{A}$ be as in Theorem 41 and suppose that $\mathcal{A}$ contains the compact operators. Then every derivation $D : \mathcal{A} \to \mathcal{A}$ is bounded, there is an operator $T \in L(X)$ such that $D(A) = TA - AT$, and D maps into $\ker(\mathcal{T}_\mathcal{A})$.*

Note that if $\mathcal{A}$ contains the compact operators then $\mathsf{TL}(X)$ contains the compact operators, which happens if and only if $V_s \to 0$ and $R_s \to 0$ weakly as $s \to \infty$ by Theorem 10.

Didas [7] derived Corollary 42 in the context of (concrete) Toeplitz and Hankel operators acting on the Hardy space H^2 on the unit disk (as defined in the introduction). Moreover, assuming that $\mathcal{C}$ is an inner subalgebra of $L^\infty(\mathbb{T})$ which strictly contains H^∞, he showed that every derivation on $\operatorname{alg} TH(\mathcal{C})$ maps into the *commutator* ideal of that algebra. The strict containment of H^∞ in $\mathcal{C}$ implies that $\operatorname{alg} TH(\mathcal{C})$ contains the compact operators and that the commutator ideal equals $\operatorname{alg} TH(\mathcal{C}) \cap \ker\mathcal{T}$ (see [8]).

If $1 < p < \infty$ and $\mathcal{S}$ is a closed subalgebra of the symbol algebra $L^\infty(\mathbb{T})$ which contains $C(\mathbb{T})$, then the Toeplitz algebra $\operatorname{alg}\mathsf{T}(\mathcal{S}) \subset L(H^p)$ contains all compact operators. An analogous result holds for concrete Toeplitz algebras on l^p and $L^p(\mathbb{R})$, $1 < p < \infty$, if $\mathcal{S}$ is a closed subalgebra of the corresponding multiplier algebras which contains all functions in the Wiener algebra over $\mathbb{T}$ and $\mathbb{R}$, respectively.

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