# On the local boundedness of weak solutions to elliptic equations with divergence-free drifts

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January 16, 2017

**Abstract**

In this paper we investigate the local boundedness of weak solutions to the equation $-\Delta u + b\cdot\nabla u = 0$ describing the diffusion in a stationary incompressible flow. The corresponding theory is well-known in the case of the general (not necessary divergence-free) sufficiently smooth drift (namely, for $b \in L_n$, where $n$ is the dimension of the space). Our main interest is focused on the case of $b$ with limited regularity (namely, $b \in L_2$). In this case the structure assumption $\operatorname{div} b = 0$ turns out to be crucial. In our paper (which is partly expository) we recall some known properties of weak solution in the case of the divergence-free drifts $b \in L_2$ and also establish some new results on the local boundedness of weak solutions.

## 1 Introduction and Main Results

Assume $n \geq 2$, $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain, $b : \Omega \to \mathbb{R}^n$, $f : \Omega \to \mathbb{R}$. In this paper we investigate the properties of weak solutions $u : \Omega \to \mathbb{R}$ to the following scalar equation

$$-\Delta u + b \cdot \nabla u \;=\; f \qquad \text{in} \quad \Omega. \tag{1.1}$$

If it is not stated otherwise, we always impose the following conditions

$$b \in L_2(\Omega) \tag{1.2}$$

$$f \in W_2^{-1}(\Omega) \tag{1.3}$$

(See the list of notation at the end of this section). We use the following definition for weak solutions:

**Definition 1.1** *Assume conditions* (1.2), (1.3) *are satisfied. The function $u \in W_2^1(\Omega)$ is called a weak solution to the equation* (1.1) *if the following integral identity holds:*

$$\int\limits_\Omega \nabla u \cdot (\nabla \eta + b\eta)\, dx \;=\; \langle f, \eta\rangle, \qquad \forall\, \eta \in C_0^\infty(\Omega). \tag{1.4}$$

[*]Both authors are supported by RFBR grant 14-01-00306.

[†]The research leading to these results has received funding from the People Programme (Marie Curie Actions) of the European Union's Seventh Framework Programme FP7/2007-2013/ under REA grant agreement n 319012 and from the Funds for International Co-operation under Polish Ministry of Science and Higher Education grant agreement n 2853/7.PR/2013/2. The author also thanks the Technische Universität of Darmstadt for its hospitality.

Together with the equation (1.1) one can consider the formally conjugate (up to the sign of the drift) equation

$$-\Delta u + \operatorname{div}(bu) \;=\; f \qquad \text{in} \quad \Omega. \tag{1.5}$$

**Definition 1.2** *Assume conditions* (1.2), (1.3) *are satisfied. The function* $u \in W_2^1(\Omega)$ *is called a weak solution to the equation* (1.5) *if*

$$\int\limits_\Omega (\nabla u - bu)\cdot \nabla \eta \; dx \;=\; \langle f, \eta\rangle, \qquad \forall\, \eta \in C_0^\infty(\Omega). \tag{1.6}$$

The advantage of the equation (1.5) is that it allows one to define weak solutions from the class $W_2^1(\Omega)$ for a drift $b$ belonging to a weaker class than (1.2). Namely, Definition 1.2 makes sense for $u \in W_2^1(\Omega)$ even if

$$b \in L_s(\Omega) \qquad \text{where} \qquad s \;=\; \left\{ \begin{array}{ll} \frac{2n}{n+2}, & n \geq 3 \\ 1+\varepsilon, \ \varepsilon > 0, & n = 2 \end{array} \right. \tag{1.7}$$

Nevertheless, for a divergence-free drift $b \in L_2(\Omega)$ the Definitions 1.1 and 1.2 coincide:

**Theorem 1.1** *Assume conditions* (1.2), (1.3) *and assume additionally*

$$\operatorname{div} b = 0 \quad \textit{in} \quad \mathcal{D}'(\Omega) \tag{1.8}$$

*Then* $u \in W_2^1(\Omega)$ *is a weak solution to the equation* (1.1) *(in the sense of Definition 1.1) if and only if it is a weak solution to* (1.5) *(in the sense of Definition 1.2).*

The proof of this result is just an integration by parts.

Together with the equation (1.1) we discuss boundary value problems with Dirichlet boundary conditions:

$$\left\{ \begin{array}{c} -\Delta u + b\cdot \nabla u = f \quad \text{in} \quad \Omega \\ u|_{\partial\Omega} = \varphi \end{array} \right. \tag{1.9}$$

For weak solutions the boundary condition is understood in the sense of traces. We distinguish between two cases: the case of "general" boundary data

$$\varphi \in W_2^{1/2}(\partial\Omega) \tag{1.10}$$

and the case of boundary data possessing some additional regularity:

$$\varphi \in L_\infty(\partial\Omega) \cap W_2^{1/2}(\partial\Omega) \tag{1.11}$$

Discussing the questions of existence and uniqueness of weak solutions to the problem (1.9) we also distinguish between another two cases: first we consider sufficiently smooth drifts, namely,

$$b \in L_n(\Omega) \tag{1.12}$$

and then we focus on the case of drifts $b$ satisfying only (1.2).

Assuming $b$ satisfies (1.2) and $u \in W_2^1(\Omega)$, $\eta \in C_0^\infty(\Omega)$ we can introduce the following quadratic form:

$$[u, \eta] \;:=\; \int\limits_\Omega \nabla u \cdot b \; \eta \; dx$$

We say this form is *bounded* on $W_2^1(\Omega)$ if there is a constant $c_b$ (depending on $b$) such that the following estimate holds:

$$\big|[u,\eta]\big| \;\leq\; c_b \, \|u\|_{W_2^1(\Omega)} \, \|\eta\|_{W_2^1(\Omega)}, \qquad \forall\, u \in W_2^1(\Omega), \qquad \forall\, \eta \in C_0^\infty(\Omega).$$

We have the following theorem:

**Theorem 1.2** *Assume $b$ satisfies* (1.12). *Then*

1) *if $n \geq 3$ then the quadratic form $[u, \eta]$ is bounded on $W^1_2(\Omega)$;*

2) *if $n = 2$ then there exist $b \in L_2(\Omega)$, $u \in \overset{\circ}{W}{}^1_2(\Omega)$ and $\eta \in \overset{\circ}{W}{}^1_2(\Omega)$ such that $\big|[u, \eta]\big| = +\infty$.*

The part 1) of Theorem 1.2 follows by the imbedding theorem and the Hölder inequality. For the part 2) one can take $\Omega = B_{1/3}$ and

$$u(x) = \eta(x) = \big|\ln|x|\big|^{1/4} - (\ln 3)^{1/4}, \qquad b(x) = \frac{-x}{|x|^2\,\big|\ln|x|\big|^{1/2}\ln\big|\ln|x|\big|}.$$

What comes to the rescue in the case of $n = 2$ is the divergence-free condition (1.8):

**Theorem 1.3** *Assume $n = 2$ and $b$ satisfies* (1.2) *and* (1.8). *Then the quadratic form $[u, \eta]$ is bounded on $W^1_2(\Omega)$.*

The proof of Theorem 1.3 is based on the application of the known div–curl lemma and the duality between $BMO$ and the Hardy space, see details in [5].

For $b$ such that the quadratic form $[u, \eta]$ is bounded on $W^1_2(\Omega)$ the existence of a weak solution to the problem (1.9) with $\varphi \equiv 0$ follows by simple application of the Lax–Millgram lemma and Fredholm theorem (i.e. existence follows from the uniqueness). In particular, in the case of $b$ satisfying (1.12), (1.8) the quadratic form $[u, \eta]$ is skew-symmetric on $\overset{\circ}{W}{}^1_2(\Omega)$ and hence

$$[u, u] = 0, \qquad \forall\ u \in \overset{\circ}{W}{}^1_2(\Omega).$$

This implies the uniqueness of weak solutions for the problem (1.9). In the case of $\varphi$ satisfying to (1.10) the problem (1.9) with non-homogeneous boundary conditions can be reduced to the corresponding problem with homogeneous boundary conditions for the function $v := u - \tilde{\varphi}$ where $\tilde{\varphi}$ is some extension of $\varphi$ from $\partial\Omega$ to $\Omega$ with the control of the norm

$$\|\tilde{\varphi}\|_{W^1_2(\Omega)} \leq c\|\varphi\|_{W^{1/2}_2(\partial\Omega)}.$$

Namely, the function $v$ can be determined as a weak solution to the problem

$$\left\{\begin{array}{c} -\Delta v + b\cdot\nabla v = f + f_\varphi \quad \text{in} \quad \Omega \\ v|_{\partial\Omega} = 0 \end{array}\right. \tag{1.13}$$

where

$$f_\varphi \ := \ \operatorname{div}\big(\nabla\tilde{\varphi} - b\tilde{\varphi}\big). \tag{1.14}$$

Note that for $n \geq 3$, $b$ satisfying (1.12) and $\tilde{\varphi} \in W^1_2(\Omega)$ we have $b\tilde{\varphi} \in L_2(\Omega)$ and hence $f_\varphi \in W^{-1}_2(\Omega)$. In the case of $n = 2$ and $b$ satisfying (1.2), (1.8) the function $b\cdot\nabla\tilde{\varphi}$ belongs to the Hardy space and hence again we have $f_\varphi \in W^{-1}_2(\Omega)$. So, we obtain the following result:

**Theorem 1.4** *Assume $b$ satisfies* (1.12), (1.8) *and $\varphi$ satisfies* (1.10). *Then for any $f$ satisfying* (1.3) *the problem* (1.9) *has the unique weak solution $u \in W^1_2(\Omega)$. Moreover, if $\varphi \equiv 0$ then this solution satisfies the energy identity*

$$\int\limits_\Omega |\nabla u|^2\, dx \ = \ \langle f, u\rangle. \tag{1.15}$$

In the case of $b$ satisfying (1.7), (1.8) we can prove the existence of at least one weak solution to the problem (1.9) if we impose some additional regularity on the boundary data $\varphi$. Namely, in this case we require $\varphi$ to be the trace of some function $\tilde{\varphi} \in W^1_2(\Omega) \cap L_\infty(\Omega)$. In this case the existence of at least one weak solution to the problem (1.9) (understood in the sense of Definition 1.2) can be easily obtained by approximations of $b$ by smooth divergence–free vector fields. Weak solutions which are weak limits (in $W^1_2(\Omega)$–norm) of solutions to a regularized problem are called *approximative weak solutions*. The uniqueness of weak solutions to the problem (1.9) for divergence-free drifts $b$ belonging to the class (1.7) does not hold even in the case of $\varphi \equiv 0$. Namely, in [10] it was constructed $b$ satisfying (1.8), (1.7) (actually a little bit better than (1.7)) and a weak solution $u \in \overset{\circ}{W}{}^1_2(\Omega)$ to the problem (1.9) with $\varphi \equiv 0$ which is not an approximative one (which immediately implies non-uniqueness, see [10] for details).

Nevertheless, if we assume some better integrability on $b$, we obtain the uniqueness of weak solutions to the problem (1.9). Namely, the following theorem holds (see [10], Theorem 1.2):

**Theorem 1.5** *Assume $f$ satisfies* (1.3)*, $b$ satisfies* (1.2)*,* (1.8) *and $\varphi$ satisfies* (1.10)*. Then a weak solution to the problem* (1.9) *is unique.*

Theorem 1.5 together with the existence of weak solutions are resumed in the following theorem:

**Theorem 1.6** *Assume $b$ satisfies* (1.2)*,* (1.8)*. Then for any $f$ satisfying* (1.3) *and any $\varphi$ satisfying* (1.11) *the problem* (1.9) *has the unique weak solution $u \in W^1_2(\Omega)$. Moreover, if $\varphi \equiv 0$ then this solution satisfies the energy inequality*

$$\int\limits_\Omega |\nabla u|^2 \, dx \ \le \ \langle f, u \rangle. \tag{1.16}$$

We prove Theorem 1.6 in Section 2. Theorems 1.4 and 1.6 demonstrate two different cases of weak solvability of the problem (1.9). Indeed, in Theorem 1.4 we assume that the boundary data are general (i.e. $\varphi$ satisfies only (1.10)), but the drift $b$ possesses some additional regularity (namely, $b$ satisfies (1.12)). On the other hand, in Theorem 1.6 the drift $b$ is an arbitrary function satisfying (1.2) while some additional regularity is required for the boundary data (i.e. $\varphi$ satisfies (1.11) instead of (1.10)).

Another important tool in the investigation of the problem (1.9) is the maximum principle. Indeed, the following theorem is true:

**Theorem 1.7** *Assume conditions* (1.2)*,* (1.8)*,* (1.11) *hold. Assume also*

$$f \in L_q(\Omega), \qquad \text{with} \qquad q > \frac{n}{2}. \tag{1.17}$$

*Let $u \in W^1_2(\Omega)$ be a weak solution to the problem* (1.9)*. Then $u \in L_\infty(\Omega)$ and*

$$\|u\|_{L_\infty(\Omega)} \ \le \ C \left( \|\varphi\|_{L_\infty(\partial\Omega)} + \|f\|_{L_q(\Omega)} \right) \tag{1.18}$$

*where the constant $C$ depend only on $n$, $q$ and $\Omega$. Moreover, if $f \equiv 0$ then*

$$\|u\|_{L_\infty(\Omega)} \ \le \ \|\varphi\|_{L_\infty(\partial\Omega)}.$$

We prove Theorem 1.7 in the Section 2.

The main goal of the present paper is to overview the *local* properties (such that local boundedness, continuity, Hölder continuity etc) of weak solutions to the equation (1.1). Note that any weak solutions to (1.1) belonging to the class $W_2^1(\Omega)$ can be viewed as a weak solution the problem (1.9) with some $\varphi$ satisfying (1.10).

We start with the discussion of the local boundedness of weak solution to (1.1). Again we distinguish between cases of drifts satisfying (1.12) and drifts satisfying only (1.2).

For $n \geq 3$ and for drifts $b$ satisfying (1.12), even not necessary satisfying (1.8), the local boundedness of weak solutions (or even their Hölder continuity) follows from the general theory, see Theorem 1.12 below. The case $n = 2$ is a bit special and, as we show in Theorem 1.8 below, this case requires the assumption (1.8). Nevertheless, if $n = 2$ and $b$ satisfies (1.2), (1.8), then weak solutions to (1.1) are locally Hölder continuous, see Theorem 1.11 below.

The following theorem shows that assumption (1.8) plays the crucial role in local boundedness of weak solutions if one considers drifts $b \in L_p(\Omega)$ with $n \geq 3$, $p < n$, or with $n = 2$, $p = 2$.

**Theorem 1.8** *Assume $\Omega = B_R$, $f \equiv 0$. Then*

1) *if $n \geq 3$ then there exist $b \in \bigcap\limits_{1 \leq p < n} L_p(\Omega)$ and a weak solution $u \in \overset{\circ}{W}{}_2^1(\Omega)$ to the problem (1.9) with $\varphi \equiv 0$ such that $u \notin L_\infty(\Omega)$;*

2) *if $n = 2$ then there exist $b \in L_2(\Omega)$ and a weak solution $\overset{\circ}{W}{}_2^1(\Omega)$ to the problem (1.9) with $\varphi \equiv 0$ such that $u \notin L_\infty(\Omega)$.*

Theorem 1.8 is proved in [2]. Note that Theorem 1.8 demonstrates also the non-uniqueness of weak solutions to the problem (1.9) with $\varphi \equiv 0$ under stated assumptions on the drift.

In the case of $b$ satisfying (1.2), (1.8) the problem of local boundedness of weak solution to (1.1) was formulated in [2] as an open question. The following two theorems (which are main results of the our paper) give the partial answer to this question:

**Theorem 1.9** *Let $B$ be a unite ball in $\mathbb{R}^n$, $n \geq 3$. Assume $b$ satisfies (1.2), (1.8) and*

$$b \in L_p(B), \qquad f \in L_p(B) \qquad \text{for some} \quad p \; > \; \frac{n}{2}. \tag{1.19}$$

*Let $u \in W_2^1(B)$ be a weak solution to (1.1). Then $u \in L_\infty(B_{1/2})$ and the following estimate holds*

$$\|u\|_{L_\infty(B_{1/2})} \; \leq \; C\, \Big( \|u\|_{W_2^1(B)} + \|f\|_{L_p(B)} \Big) \tag{1.20}$$

*where the constant $C$ depends only on $n$, $p$ and $\|b\|_{L_p(B)}$.*

Note that for $b$ satisfying (1.2) the condition $b \in L_p(B)$ with $p > \frac{n}{2}$ is superfluous if $n = 3$.

**Theorem 1.10** *Assume $n \geq 4$, $B$ is a unite ball in $\mathbb{R}^n$. Then there exist $b$ satisfying to (1.8) and*

$$b \in L_p(B) \quad \text{for any} \quad p \; < \; \frac{n-1}{2}, \tag{1.21}$$

*and a weak solution $u \in W_2^1(B)$ to (1.1) (in the sense of Definition 1.2) with $f \equiv 0$ such that*

$$u \notin L_\infty(B_{1/2})$$

*Namely,*

$$u(x) = \log r, \qquad b = (n-3) \left( \frac{1}{r}\, \mathbf{e}_r - (n-3)\, \frac{z}{r^2}\, \mathbf{e}_z \right),$$

*where $r^2 = x_1^2 + ... + x_{n-1}^2$, $z = x_n$ and $\mathbf{e}_r$, $\mathbf{e}_z$ are the basis vectors of the corresponding cylindrical coordinate system in $\mathbb{R}^n$.*

Theorem 1.9 seems to be new. We prove it in Section 3. Theorem 1.10 is also new and it can be verified by direct computations. Note, that the assumption $n \geq 4$ in Theorem 1.10 is necessary to provide $u \in W^1_2(B)$. The drift $b$ in Theorem 1.10 has the asymptotic $r^{-2}$ near the axis of symmetry and hence (1.21) holds. It is easy to see that if we want to have in Theorem 1.10 $b \in L_2(\Omega)$ then it leads to the restriction $n \geq 6$.

Theorems 1.7 and 1.10 together establish an interesting phenomena: for drifts $b$ which are not sufficiently smooth, namely, satisfy only (1.2) and (1.8), the property of the elliptic operator in (1.1) (with zero right-hand side) to improve the "regularity" of weak solutions (in the sense that every weak solution is locally bounded) in higher dimensions (i.e. for $n \geq 6$) depends on the behavior of a weak solution on the boundary of the domain. If the values of a weak solution $\varphi := u|_{\partial\Omega}$ on the boundary are bounded (i.e. $\varphi$ satisfies (1.11)) then this weak solution must be bounded (as Theorem 1.7 says). On the other hand, if the function $\varphi$ is singular on $\partial\Omega$ (but still $\varphi$ satisfies (1.10)) then the weak solution can be unbounded even near internal points of of the domain $\Omega$ (as Theorem 1.10 shows). To our opinion such a behavior of solutions to an elliptic equation is unexpected. Allowing some abuse of language we can say that in a sense *non-smoothness of the drift can destroy the hypoellipticity of the operator*.

Theorem 1.7 impose some restrictions on the structure of set of singular points of weak solutions. Namely, let us define a *singular point* of a weak solution as a point for which the weak solution is unbounded in any its neighborhood and then define the *singular set* of a weak solution as the set of all its singular points. Theorem 1.7 shows that if for some weak solution its singular set is non-empty then its 1-dimensional Hausdorff measure must be positive. Indeed, due to Theorem 1.7 a singular point, if exists, never can be surrounded by a ball with regular values on its boundary and hence the singular set must have a non-empty intersection with every sphere centered at a singular point. In particular, this means that no isolated singularity is possible. This exactly what the counterexample in Theorem 1.10 demonstrates: the singular set in this case is the axe of symmetry (i.e. the whole line).

In conclusion we remark that for the moment of writing of this paper we can say nothing about local boundedness of weak solutions to (1.1) with $b$ satisfying only (1.2), (1.8) in the case of $n = 4, 5$. We state this problem as an open question.

Now we pass to the discussion of local continuity and Hölder continuity of weak solutions to the problem (1.1). For simplicity we will assume $f \equiv 0$ in (1.1). To the contrast to the local boundedness, it turns out that the divergence-free condition (1.8) does not help in this situation too much.

We start with the 2D-case. As we know from Theorem 1.8 for $b$ satisfying (1.2) without condition (1.8) for $n = 2$ weak solutions to the equation (1.1) can be locally unbounded. Nevertheless, if we additionally assume (1.8), we immediately obtain not only local boundedness, but even local Hölder continuity:

**Theorem 1.11** *Assume $n = 2$ and let $b$ satisfy* (1.2) *and* (1.8)*. Let $u \in W^1_2(B)$ be a weak solution to* (1.1) *with $f \in L_p(B)$ for some $p > 1$. Then $u \in C^\alpha_{loc}(B)$ with any $\alpha < 1$.*

For the proof in the case of $f \equiv 0$ see [2]. The case of $f \not\equiv 0$ can be considered in a similar way. Note that for $n = 2$ the statement of Theorem 1.11 remains true if $b$, not necessary satisfying (1.8), meets the condition

$$\int\limits_B |b|^2 \ln(2 + |b|^2)\, dx \ < \ +\infty.$$

The proof can be found in [2], §4.4 or in [7].

As we mentioned before, for $n \geq 3$ and drifts $b$ satisfying (1.12) we obtain the local Hölder continuity of weak solutions even without divergence-free assumption (1.8):

**Theorem 1.12** *Assume $n \geq 3$ and let $b$ satisfy* (1.12)*. Let $u \in W_2^1(B)$ be a weak solution to* (1.1) *with $f \in L_p(B)$ for some $p > \frac{n}{2}$. Then $u \in C^\alpha_{loc}(B)$ with any $\alpha < 1$.*

For the proof see [2].

The following counterexample constructed in [2] shows that if one is interested in the local continuity of weak solutions to the equation (1.1) then the assumption $b \in L_n(\Omega)$ can not be weakened and the structure condition (1.8) does not help in this situation at all:

**Theorem 1.13** *Assume $n \geq 3$, $2 \leq p < n$. Then there exist $b \in L_p(B)$ satisfying* (1.8) *and a weak solution $u$ to* (1.1) *with $f \equiv 0$ such that $u \in W_2^1(B) \cap L_\infty(B)$ but $u \notin C(\bar{B}_{1/2})$.*

Finally, if we assume that $b$ is better than in (1.12) (even without condition (1.8)), we immediately obtain that the gradient of weak solutions is locally Hölder continuous:

**Theorem 1.14** *Assume $b \in L_p(B)$ with $p > n$ and $u \in W_2^1(B)$ is a weak solution to* (1.1) *with $f \in L_p(B)$. Then $u \in C^{1+\alpha}_{loc}(B)$ with $\alpha = 1 - \frac{n}{p}$.*

For the proof see [4], Chapter III, Theorem 15.1.

In conclusion we remark that though we consider only an elliptic equation with a drift, but there are also many papers devoted to the corresponding parabolic equation, see, for example, [8], [9] and reference there.

Our paper is organized as follows. In Section 2 we prove Theorems 1.6 and 1.7. In Section 3 we present the proof of Theorem 1.9.

In the paper we explore the following notation. For any $a$, $b \in \mathbb{R}^n$ we denote by $a \cdot b$ its scalar product in $\mathbb{R}^n$. For any $p \in [1, +\infty)$ we denote by $L_p(\Omega)$ and $W^k_p(\Omega)$ the usual Lebesgue and Sobolev spaces. We do not distinguish between spaces of scalar functions and vector fields in the notation. The space of measurable functions whose values are essentially bounded in $\Omega$ is denoted by $L_\infty(\Omega)$. We denote by $C_0^\infty(\Omega)$ the set of all smooth functions which are compactly supported in $\Omega$. The space $\overset{\circ}{W}{}^1_p(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^1_p(\Omega)$ norm. We denote by $\mathcal{D}'(\Omega)$ the set of distributions on $\Omega$. The negative Sobolev space $W^{-1}_p(\Omega)$, $p \in (1, +\infty)$, is the set of all distributions which are bounded functionals on $\overset{\circ}{W}{}^1_{p'}(\Omega)$ with $p' := \frac{p}{p-1}$. For any $f \in W^{-1}_p(\Omega)$ and $w \in \overset{\circ}{W}{}^1_{p'}(\Omega)$ we denote by $\langle f, w\rangle$ the value of the distribution $f$ on the function $w$. We use the notation $W_2^{1/2}(\partial\Omega)$ for the fractional Slobodetskii–Sobolev space which consists of traces of all functions from $W^1_2(\Omega)$. By $C(\bar{\Omega})$ and $C^\alpha(\bar{\Omega})$, $\alpha \in (0,1)$ we denote the Banach spaces of continuous and Hölder continuous functions on $\bar{\Omega}$. The space $C^{1+\alpha}(\bar{\Omega})$ consists of functions $u$ whose gradient $\nabla u$ is Hölder continuous. The index "loc" in notation of the functional spaces $L_{\infty,loc}(\Omega)$, $C^\alpha_{loc}(\Omega)$, $C^{1+\alpha}_{loc}(\Omega)$ etc implies that the function belongs to the corresponding functional class over every compact set which is contained in $\Omega$. The symbols $\rightharpoonup$ and $\rightarrow$ stand for the weak and strong convergence respectively. We denote by $B_R$ or $B(R)$ the ball in $\mathbb{R}^n$ of radius $R$ centered at the origin. We write $B$ instead of $B_1$.

## 2 Preliminaries

First we establish some regularization result.

**Theorem 2.1** *Assume $b$ satisfies* (1.2), (1.8), *$\varphi$ satisfies* (1.11) *and $f$ satisfies* (1.3). *Let $b^\varepsilon$ be a sequence of vector fields satisfying* (1.12), (1.8) *such that $b^\varepsilon \to b$ in $L_2(\Omega)$. Assume $u^\varepsilon \in W_2^1(\Omega)$ is a weak solution to the regularized problem*

$$\left\{ \begin{array}{c} -\Delta u^\varepsilon + b^\varepsilon \cdot \nabla u^\varepsilon = f \quad in \quad \Omega \\ u^\varepsilon|_{\partial\Omega} = \varphi \end{array} \right. \tag{2.1}$$

*(The existence and uniqueness of $u^\varepsilon$ is guaranteed by Theorem 1.4). Then the sequence $\{u^\varepsilon\}$ is bounded in $W_2^1(\Omega)$ and hence there is a function $u \in W_2^1(\Omega)$ and a subsequence of $\{u^\varepsilon\}$ (for which we keep the same notation) such that*

$$u^\varepsilon \rightharpoonup u \quad in \quad W_2^1(\Omega), \tag{2.2}$$

*Moreover, the function $u$ is a weak solution to the problem* (1.9) *(understood in the sense of Definition 1.2). Additionally, if $\varphi \equiv 0$ then $u$ satisfies the energy inequality* (1.16).

To prove Theorem 2.1 we need the following extension result:

**Theorem 2.2** *Assume $\Omega \subset \mathbb{R}^n$ is a bounded domain of class $C^1$. Then there exists a bounded linear extension operator $T : L_\infty(\partial\Omega) \cap W_2^{1/2}(\partial\Omega) \to L_\infty(\Omega) \cap W_2^1(\Omega)$ such that*

$$T\varphi|_{\partial\Omega} \;=\; \varphi, \qquad \forall\, \varphi \in L_\infty(\partial\Omega) \cap W_2^{1/2}(\partial\Omega).$$

$$\|T\varphi\|_{W_2^1(\Omega)} \;\le\; C(\Omega)\, \|\varphi\|_{W_2^{1/2}(\partial\Omega)}, \qquad \|T\varphi\|_{L_\infty(\Omega)} \;\le\; C(\Omega)\, \|\varphi\|_{L_\infty(\partial\Omega)}$$

**Proof of Theorem 2.2**

For the sake of completeness we briefly recall the proof of Theorem 2.2. After the localization and flattening of the boundary it is sufficient to construct the extension operator from $\mathbb{R}^{n-1}$ to $\mathbb{R}^n_+ := \mathbb{R}^{n-1} \times (0, +\infty)$. Then we can take the standard operator

$$(T\varphi)(x', x_n) \;=\; \eta(x_n) \int\limits_{\mathbb{R}^{n-1}} \varphi(x' - x_n\xi')\psi(\xi')\, d\xi', \qquad (x', x_n) \in \mathbb{R}^n_+,$$

where $x' := (x_1, \dots, x_{n-1})$, $x' \in \mathbb{R}^{n-1}$, $\eta \in C_0^\infty(\mathbb{R})$, $\psi(0) = 1$, $\psi \in C_0^\infty(\mathbb{R}^{n-1})$, $\int\limits_{\mathbb{R}^{n-1}} \psi(\xi')\, d\xi' = 1$. This operator is bounded from $W_2^{1/2}(\mathbb{R}^{n-1})$ to $W_2^1(\mathbb{R}^n_+)$ and also from $L_\infty(\mathbb{R}^{n-1})$ to $L_\infty(\mathbb{R}^n_+)$. More details can be found in [1]. $\square$

**Proof of Theorem 2.1**

Denote $\tilde\varphi := T\varphi$ where $T$ is the extension operator from Theorem 2.2. Taking in the integral identity (1.6) for $u^\varepsilon$ and $b^\varepsilon$ the test function $\eta = u^\varepsilon - \tilde\varphi \in \overset{\circ}{W}{}_2^1(\Omega)$ we obtain

$$\int\limits_\Omega |\nabla u^\varepsilon|^2\, dx \;-\; \int\limits_\Omega u^\varepsilon b^\varepsilon \cdot \nabla(u^\varepsilon - \tilde\varphi)\, dx \;=\; \langle f, u^\varepsilon - \tilde\varphi\rangle$$

Using the divergence–free condition (1.8) we obtain

$$\int\limits_{\Omega} u^\varepsilon b^\varepsilon \cdot \nabla(u^\varepsilon - \tilde{\varphi})\, dx \;=\; \int\limits_{\Omega} \tilde{\varphi} b^\varepsilon \cdot \nabla(u^\varepsilon - \tilde{\varphi})\, dx$$

Hence we obtain the estimate

$$\|\nabla u^\varepsilon\|^2_{L_2(\Omega)} \;\le\; \Big(\|\tilde{\varphi}\|_{L_\infty(\Omega)}\|b^\varepsilon\|_{L_2(\Omega)} \;+\; \|f\|_{W_2^{-1}(\Omega)}\Big)\Big(\|u^\varepsilon\|_{W_2^1(\Omega)} + \|\tilde{\varphi}\|_{W_2^1(\Omega)}\Big)$$

From this inequality we obtain the estimate

$$\|u^\varepsilon\|_{W_2^1(\Omega)} \;\le\; C,$$

with the constant $C$ independent on $\varepsilon$. The rest of the proof is obvious. $\square$

**Proof of Theorem 1.6**

The existence of at least one weak solution follows from Theorem 2.1. The uniqueness of this solution $u$ for $b$ satisfying (1.2), (1.8) is established in Theorem 1.5. Moreover, as a weak solution is unique we obtain that the whole sequence $\{u^\varepsilon\}$ in Theorem 2.1 must converge weakly in $W_2^1(\Omega)$ to the function $u$. The inequality (1.16) in the case of $\varphi \equiv 0$ follows from the energy identity (1.15) for $u^\varepsilon$ and lower semicontinuity of the norm with respect to the weak convergence. Theorem (1.6) is proved. $\square$

To prove Theorem 1.7 we first prove the corresponding result for a regularized problem:

**Theorem 2.3** *Assume conditions* (1.12), (1.8), (1.11), (1.17) *hold. Let* $u \in W_2^1(\Omega)$ *be a weak solution to the problem* (1.9). *Then* $u \in L_\infty(\Omega)$ *and the estimate* (1.18) *holds with the constant* $C$ *depending only on* $n$, $q$ *and* $\Omega$.

**Proof of Theorem 2.3**

We present the proof only in the case $n \ge 3$. The case $n = 2$ differs from it only by routine technical details. As $b$ satisfies (1.12) we can complete the integral identity (1.4) up to the test functions $\eta \in \overset{\circ}{W}{}_2^1(\Omega)$. Denote $k_0 := \|\varphi\|_{L_\infty(\partial\Omega)}$ and assume $k \ge k_0$. Take in (1.4) $\eta = (u-k)_+$, where we denote $(u)_+ := \max\{u, 0\}$. As $k \ge k_0$ we have $\eta \in \overset{\circ}{W}{}_2^1(\Omega)$ and $\nabla\eta = \chi_{A_k}\nabla u$ where $\chi_{A_k}$ is the characteristic function of the set

$$A_k \;:=\; \{\; x \in \Omega : \; u(x) > k \;\}.$$

We obtain the identity

$$\int\limits_{A_k} |\nabla u|^2\, dx \;+\; \int\limits_{A_k} b\cdot(u-k)\nabla u\, dx \;=\; \int\limits_{A_k} f(u-k)\, dx$$

From (1.8) we obtain

$$\int\limits_{A_k} b\cdot(u-k)\nabla u\, dx \;=\; \frac{1}{2}\int\limits_{\Omega} b\cdot\nabla|(u-k)_+|^2\, dx \;=\; 0,$$

and hence

$$\int\limits_{A_k} |\nabla u|^2\, dx \;=\; \int\limits_{A_k} f(u-k)\, dx, \qquad \forall\; k \ge k_0.$$

The rest of the proof goes as in the usual elliptic theory: applying the imbedding theorem we obtain

$$\left( \int\limits_{A_k} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} \ \le \ C(n) \ \left( \int\limits_{A_k} |f|^{\frac{2n}{n+2}} \, dx \right)^{\frac{n+2}{2n}},$$

and using the Hölder inequality we get

$$\|f\|_{L_{\frac{2n}{n+2}}(A_k)} \ \le \ |A_k|^{\frac{n+2}{2n}-\frac{1}{q}} \ \|f\|_{L_q(A_k)}$$

So we arrive at

$$\int\limits_{A_k} |\nabla u|^2 \, dx \ \le \ C(n) \ \|f\|^2_{L_q(\Omega)} \ |A_k|^{1-\frac{2}{n}+\varepsilon}, \qquad \forall \ k \ge k_0,$$

where $\varepsilon := 2\left(\frac{2}{n}-\frac{1}{q}\right) > 0$. This inequality implies the following estimate, see [4], Lemma 5.3:

$$\operatorname*{esssup}_{\Omega}(u-k_0)_+ \ \le \ C(n,q,\Omega) \ \|f\|_{L_q(\Omega)}$$

The estimate of $\operatorname*{essinf}_{\Omega} u$ can be obtained in a similar way if we replace $u$ by $-u$. Theorem 2.3 is proved. $\square$

**Proof of Theorem 1.7**

Assume $u$ is a weak solution to the problem (1.9) with $b$ satisfying (1.2), (1.8) and let $b^\varepsilon$ be smooth divergence-free vector fields such that $b^\varepsilon \to b$ in $L_2(\Omega)$. Denote by $u^\varepsilon$ the weak solution to the problem (2.1). Then from Theorem 2.3 we obtain the estimate

$$\|u^\varepsilon\|_{L_\infty(\Omega)} \ \le \ C \ \Big( \|\varphi\|_{L_\infty(\partial\Omega)} + \|f\|_{L_q(\Omega)} \Big) \tag{2.3}$$

with the constant $C$ depending only on $n$, $q$, $\Omega$. From (2.2) we can extract a subsequence (for which we keep the same notation) such that

$$u^\varepsilon \to u \quad \text{a.e. in} \quad \Omega.$$

This subsequence is uniformly bounded and passing to the limit in (2.3) we obtain (1.18) for the unique weak solution $u$ to the problem (1.9). Theorem 1.7 is proved. $\square$

# 3 Proof of Theorem 1.9

First we derive the estimate (1.20) as an a priori estimate under additional assumption that the weak solution is bounded. We explore Moser's iteration technique, see [6], see also [3].

**Theorem 3.1** *Assume all conditions of Theorem 1.9 hold and assume additionally that* $u \in L_\infty(B)$. *Then the estimate*

$$\|u\|_{L_\infty(B(\frac12))} \ \le \ C \ \left(2+\|b\|_{L_p(B)}\right)^\mu \ \left(\|u\|_{L_{2p'}(B)} + \|f\|_{L_p(B)}\right) \tag{3.1}$$

*holds with some positive constants* $C$ *and* $\mu$ *depending only on* $n$ *and* $p$.

**Proof of Theorem 3.1**

Let $\zeta \in C_0^\infty(B)$ be a cut-off function such that $\zeta \ge 0$, $0 \le \zeta \le 1$ and assume $\beta \ge 0$ and $k \ge 0$ are arbitrary. Denote

$$\bar u \ := \ \max\{u,0\} \ + \ k, \qquad A \ := \ \{ \ x\in B: \ u(x)>0 \ \},$$

and take in (1.4) the test function

$$\eta \ = \ \zeta^2\left(\bar u^{\beta+1}-k^{\beta+1}\right) \ \in \ \overset{\circ}{W}{}^1_2(B)\cap L_\infty(B).$$

We have

$$\nabla\eta \ = \ (\beta+1)\zeta^2\bar u^\beta\nabla\bar u \ + \ 2\zeta\nabla\zeta\left(\bar u^{\beta+1}-k^{\beta+1}\right)$$

and as $\nabla\bar u = \nabla u$ a.e. in $A$ and $\eta\equiv 0$ on $B\setminus A$ we obtain

$$\int\limits_B \nabla u\cdot\nabla\eta\, dx \ = \ \int\limits_A \nabla\bar u\cdot\nabla\eta\, dx \ = \ (\beta+1)\int\limits_B \zeta^2\bar u^\beta|\nabla\bar u|^2\, dx \ + \ 2\int\limits_B \zeta\nabla\bar u\cdot(\bar u^{\beta+1}-k^{\beta+1})\nabla\zeta\, dx$$

$$\int\limits_B \zeta\nabla\bar u\cdot(\bar u^{\beta+1}-k^{\beta+1})\nabla\zeta\, dx \ = \ \int\limits_B \nabla\bar u\cdot\bar u^{\beta+1}\zeta\nabla\zeta\, dx \ - \ k^{\beta+1}\int\limits_B \nabla\bar u\cdot\zeta\nabla\zeta\, dx \ =$$

$$= \ -\frac{1}{\beta+2}\int\limits_B \bar u^{\beta+2}\operatorname{div}(\zeta\nabla\zeta)\, dx \ + \ k^{\beta+1}\int\limits_B \bar u\operatorname{div}(\zeta\nabla\zeta)\, dx$$

The drift term is estimated in the following way:

$$\int\limits_B b\cdot\nabla u\ \eta\, dx \ = \ \int\limits_A b\cdot\nabla\bar u\ \eta\, dx \ = \ \frac{1}{\beta+2}\int\limits_B \zeta^2 b\cdot\nabla\bar u^{\beta+2}\, dx \ - \ k^{\beta+1}\int\limits_B \zeta^2 b\cdot\nabla\bar u\, dx$$

Integrating by parts we obtain

$$\int\limits_B b\cdot\nabla u\ \eta\, dx \ = \ -\frac{2}{\beta+2}\int\limits_B \zeta\nabla\zeta\cdot b\ \bar u^{\beta+2}\, dx \ + \ 2k^{\beta+1}\int\limits_B \zeta\nabla\zeta\cdot b\bar u\, dx$$

So, we obtain the identity

$$(\beta+1)\int\limits_B \zeta^2\bar u^\beta|\nabla\bar u|^2\, dx \ = \ \frac{2}{\beta+2}\int\limits_B \bar u^{\beta+2}\operatorname{div}(\zeta\nabla\zeta)\, dx \ - \ 2k^{\beta+1}\int\limits_B \bar u\operatorname{div}(\zeta\nabla\zeta)\, dx \ +$$

$$+ \ \frac{2}{\beta+2}\int\limits_B \zeta\nabla\zeta\cdot b\ \bar u^{\beta+2}\, dx \ - \ 2k^{\beta+1}\int\limits_B \zeta\nabla\zeta\cdot b\bar u\, dx \ + \ \int\limits_B \zeta^2 f\left(\bar u^{\beta+1}-k^{\beta+1}\right)\, dx$$

Hence we obtain the inequality

$$\int_B \zeta^2\bar{u}^\beta|\nabla\bar{u}|^2\,dx \;\le\; \frac{2}{(\beta+1)^2}\int_B \bar{u}^{\beta+2}\Big(|\nabla\zeta|^2+|\nabla^2\zeta|+|b|\,|\nabla\zeta|\Big)\,dx\,+$$
$$+\;\frac{2k^{\beta+1}}{\beta+1}\int_B \bar{u}\Big(|\nabla\zeta|^2+|\nabla^2\zeta|+|b|\,|\nabla\zeta|\Big)\,dx\;+\;\frac{1}{\beta+1}\int_B |f|\;\bar{u}^{\beta+1}\,dx$$

Taking into account that $k\le\bar{u}$ and $\frac{1}{(\beta+1)^2}\le\frac{1}{\beta+1}$ we obtain

$$\int_B \zeta^2\bar{u}^\beta|\nabla\bar{u}|^2\,dx \;\le\; \frac{4}{\beta+1}\int_B \bar{u}^{\beta+2}\Big(|\nabla\zeta|^2+|\nabla^2\zeta|+|b|\,|\nabla\zeta|\Big)\,dx\;+\;\frac{1}{\beta+1}\int_B |f|\;\bar{u}^{\beta+1}\,dx$$

If $k>0$ the last integral we estimate as follows:

$$\int_B |f|\;\bar{u}^{\beta+1}\,dx\;\le\;\Big\|\frac{f}{k}\Big\|_{L_q(B)}\Big\|\bar{u}^{\beta+2}\Big\|_{L_{q'}(B)}$$

Now fix $k:=\|f\|_{L_q(B)}$ if $f\not\equiv 0$ and $k=0$ if $f\equiv 0$. In both cases we obtain

$$\int_B \zeta^2\bar{u}^\beta|\nabla\bar{u}|^2\,dx \;\le\; \frac{4}{\beta+1}\int_B \bar{u}^{\beta+2}\Big(|\nabla\zeta|^2+|\nabla^2\zeta|+|b|\,|\nabla\zeta|\Big)\,dx\;+\;\frac{1}{\beta+1}\Big\|\bar{u}^{\beta+2}\Big\|_{L_{q'}(B)}$$

Denote by

$$w\;:=\;\bar{u}^{\frac{\beta+2}{2}}\quad\Longrightarrow\quad\nabla w=\frac{\beta+2}{2}\;\bar{u}^{\frac{\beta}{2}}\nabla\bar{u},\quad|\nabla w|^2=\frac{(\beta+2)^2}{4}\;\bar{u}^\beta|\nabla\bar{u}|^2,\quad\bar{u}^{\beta+2}=w^2,$$

Hence we obtain the estimate

$$\frac{4}{(\beta+2)^2}\int_B \zeta^2|\nabla w|^2\,dx\;\le\;\frac{4}{\beta+1}\int_B |w|^2\;\Big(|\nabla\zeta|^2+|\nabla^2\zeta|+|b|\,|\nabla\zeta|\Big)\,dx\;+\;\frac{1}{\beta+1}\;\|w\|^2_{L_{2q'}(B)}$$

which gives

$$\int_B |\nabla(\zeta w)|^2\,dx\;\le\;C(\beta+2)\;\left(\int_B |w|^2\;\Big(|\nabla\zeta|^2+|\nabla^2\zeta|+|b|\,|\nabla\zeta|\Big)\,dx\;+\;\|w\|^2_{L_{2q'}(B)}\right)$$

Applying the imbedding theorem we obtain

$$\left(\int_B |\zeta w|^{\frac{2n}{n-2}}\,dx\right)^{\frac{n-2}{n}}\;\le\;C\int_B |\nabla(\zeta w)|^2\,dx$$

Hence

$$\left(\int_B |\zeta w|^{\frac{2n}{n-2}}\,dx\right)^{\frac{n-2}{n}}\;\le\;C(\beta+2)\left(\int_B |w|^2\;\Big(|\nabla\zeta|^2+|\nabla^2\zeta|+|b|\,|\nabla\zeta|\Big)\,dx\;+\;\|w\|^2_{L_{2q'}(B)}\right)$$

Assume $\zeta\equiv 1$ on $B_r$, $\zeta\in C_0^\infty(B_R)$ and $|\nabla\zeta|\le\frac{C}{R-r}$, $|\nabla^2\zeta|\le\frac{C}{(R-r)^2}$. Applying the Hölder inequality we obtain

$$\int_B |w|^2|b|\,|\nabla\zeta|\,dx\;\le\;\frac{C}{R-r}\;\|b\|_{L_p(B_R)}\;\|w\|^2_{L_{2p'}(B_R)}$$

where $p' := \frac{p}{p-1}$. As $p > \frac{n}{2}$ we have $\frac{2n}{n-2} > 2p'$. (Without loss of generality we can take $p = q$). So, we obtain

$$\|w\|_{L_{\frac{2n}{n-2}}(B_r)} \;\leq\; \frac{C\sqrt{\beta+2}}{R-r}\, \|w\|_{L_2(B_R)} \;+\; \sqrt{\beta+2}\left(\frac{C}{(R-r)^{1/2}}\, \|b\|^{\frac{1}{2}}_{L_p(B_R)} + 1\right)\, \|w\|_{L_{2p'}(B_R)}$$

As $2 \leq 2p'$ by the Hölder inequality we have

$$\|w\|_{L_2(B_R)} \;\leq\; C\, \|w\|_{L_{2p'}(B_R)}$$

Clearly,

$$\frac{C}{(R-r)^{1/2}}\, \|b\|^{\frac{1}{2}}_{L_p(B_R)} \;\leq\; \frac{C}{R-r} \;+\; \|b\|_{L_p(B_R)}$$

and hence

$$\|w\|_{L_{\frac{2n}{n-2}}(B_r)} \;\leq\; \sqrt{\beta+2}\left(\frac{C_0}{R-r} \;+\; \|b\|_{L_p(B_R)} + 1\right)\, \|w\|_{L_{2p'}(B_R)}$$

Going back to the function $\bar{u}$ and denoting $\beta + 2 =: \gamma$ we obtain

$$\|\bar{u}\|_{L_{\frac{n\gamma}{n-2}}(B_r)} \;\leq\; e^{\frac{\ln\gamma}{\gamma}}\left(\frac{C_0}{R-r} + \|b\|_{L_p(B_R)} + 1\right)^{\frac{2}{\gamma}}\, \|\bar{u}\|_{L_{p'\gamma}(B_R)} \tag{3.2}$$

Denote $\chi := \frac{1}{p'}\frac{n}{n-2} = \frac{n(p-1)}{p(n-2)}$, $s_0 = 2p'$, $s_m := \chi s_{m-1} = 2p'\chi^m$, $R_m = \frac{1}{2} + \frac{1}{2^{m+1}}$. Then

$$\|\bar{u}\|_{L_{s_m}(B_{R_m})} \;=\; \|\bar{u}\|_{L_{\frac{n}{n-2}\frac{s_{m-1}}{p'}}(B_{R_m})}$$

Applying (3.2) with $\gamma = \frac{s_{m-1}}{p'}$, $r = R_m$, $R = R_{m-1}$, we obtain the inequality

$$\|\bar{u}\|_{L_{s_m}(B_{R_m})} \;\leq\; e^{\frac{\ln(2\chi^{m-1})}{2\chi^{m-1}}}\left(\frac{C_0}{R_{m-1}-R_m} \;+\; \|b\|_{L_p(B)} + 1\right)^{\frac{1}{\chi^{m-1}}}\, \|\bar{u}\|_{L_{s_{m-1}}(B_{R_{m-1}})} \;=$$
$$=\; e^{\frac{\ln 2 + (m-1)\ln\chi}{2\chi^{m-1}}}\left(C_0 2^{m+1} \;+\; \|b\|_{L_p(B)} + 1\right)^{\frac{1}{\chi^{m-1}}}\, \|\bar{u}\|_{L_{s_{m-1}}(B_{R_{m-1}})}$$

Iterating this estimate we obtain

$$\|\bar{u}\|_{L_{s_m}(B_{R_m})} \;\leq\; \prod_{j=1}^{m} e^{\frac{\ln 2 + (j-1)\ln\chi}{2\chi^{j-1}}}\left(C_0 2^{j+1} + \|b\|_{L_p(B)} + 1\right)^{\frac{1}{\chi^{j-1}}}\, \|\bar{u}\|_{L_{s_0}(B)}$$

Note that

$$\prod_{j=1}^{m} e^{\frac{\ln 2 + (j-1)\ln\chi}{2\chi^{j-1}}}\left(C_0 2^{j+1} + \|b\|_{L_p(B)} + 1\right)^{\frac{1}{\chi^{j-1}}} \;=$$
$$=\; \exp\left[\sum_{j=1}^{m}\frac{\ln 2 + (j-1)\ln\chi}{2\chi^{j-1}}\right]\cdot \exp\left[\sum_{j=1}^{m}\frac{1}{\chi^{j-1}}\;\ln\left(C_0 2^{j+1} + \|b\|_{L_p(B)} + 1\right)\right]$$

The following series is convergent:

$$\sum_{j=1}^{\infty}\frac{\ln 2 + (j-1)\ln\chi}{\chi^{j-1}} \;<\; +\infty$$

As for $x \geq 2$, $y \geq 2$ the inequality $\ln(x+y) \leq \ln x + \ln y$ is true, we have

$$\ln\left(C_0 2^{j+1} + \|b\|_{L_p(B)} + 1\right) \;\leq\; \ln(C_0 2^{j+1}) \;+\; \ln(2 + \|b\|_{L_p(B)})$$

and we have one more convergent series:

$$\sum_{j=0}^{\infty} \frac{1}{\chi^j} \, \Big( \ln(C_0 2^{j+2}) + \ln(2 + \|b\|_{L_p(B)}) \Big) \; < \; +\infty.$$

Hence for any $m = 0, 1, 2, \dots$ the following estimate holds

$$\|\bar u\|_{L_{s_m}(B(\frac12))} \; \le \; C \; \big(2 + \|b\|_{L_p(B)}\big)^{\mu} \; \|u\|_{L_{2p'}(B)}$$

where $C$ and $\mu$ depend only on $n$ and $p$. Taking the limit as $m \to +\infty$ we obtain

$$\|\bar u\|_{L_\infty(B(\frac12))} \; \le \; C \; \big(2 + \|b\|_{L_p(B)}\big)^{\mu} \; \|\bar u\|_{L_{2p'}(B)}$$

The last inequality implies the estimate

$$\|u_+\|_{L_\infty(B(\frac12))} \; \le \; C \; \big(2 + \|b\|_{L_p(B)}\big)^{\mu} \; \Big(\|u\|_{L_{2p'}(B)} + k\Big)$$

with some positive constant $C$ and $\mu$ depending only on $n$ and $p$. The estimate of $\|u_-\|_{L_\infty(B(\frac12))}$ where $u_- := \max\{-u, 0\}$ can be obtained in a similar way. Theorem 3.1 is proved. □

**Proof of Theorem 1.9.**

We need to get rid of the assumption of $u \in L_\infty(B)$ in Theorem 3.1. Let all assumptions of Theorem 1.9 hold. Assume $\zeta \in C_0^\infty(B)$ is a cut-off function such that $\zeta \equiv 1$ on $B(\frac56)$ and denote $v := \zeta u$. Then $v$ is a weak solution to the boundary value problem

$$\left\{ \begin{array}{c} -\Delta v + b\cdot \nabla v = g \quad \text{in} \quad B \\ v|_{\partial B} = 0 \end{array}\right.$$

where

$$g := \zeta f + f_\zeta + f_b, \qquad f_\zeta := -u\Delta\zeta - 2\nabla u\cdot\nabla\zeta, \qquad f_b := bu\cdot\nabla\zeta$$

Note that both $f_\zeta \equiv f_b \equiv 0$ on $B(\frac56)$, $g \equiv f$ and $v \equiv u$ on $B(\frac56)$. As $p > \frac n2$ we have

$$f_\zeta \in L_2(B), \qquad f_b \in L_s(B), \quad \text{with some} \quad s > \frac{2n}{n+2},$$

and hence we have $g \in W_2^{-1}(B)$. Assume now $b^\varepsilon$ be a sequence of smooth divergence-free vector fields such that $b^\varepsilon \to b$ in $L_p(B)$ and let $v^\varepsilon \in \overset{\circ}{W}{}^1_2(B)$ be the unique weak solution to the boundary value problem

$$\left\{ \begin{array}{c} -\Delta v^\varepsilon + b^\varepsilon\cdot \nabla v^\varepsilon = g \quad \text{in} \quad B \\ v^\varepsilon|_{\partial B} = 0 \end{array}\right.$$

From Theorem 2.1 we have $v^\varepsilon \rightharpoonup v$ in $W^1_2(B)$ and as $p > \frac n2$ we can extract a subsequence (for which we keep the same notation) such that $v^\varepsilon \to v$ a.e. in $B$ and $v^\varepsilon \to v$ in $L_{2p'}(B)$. As $v^\varepsilon$ is a weak solution to the equation

$$-\Delta v^\varepsilon + b^\varepsilon\cdot\nabla v^\varepsilon \; = \; f \qquad \text{in} \qquad B(\tfrac56)$$

with smooth $b^\varepsilon$ and $f \in L_p(B)$, $p > \frac n2$, from the usual elliptic theory (see [4], Chapter III, Theorem 14.1) we conclude that $v^\varepsilon$ is Hölder continuous in $B(\frac34)$. Hence applying Theorem 3.1 (with the obvious modification in radius) we obtain the estimate

$$\|v^\varepsilon\|_{L_\infty(B(\frac12))} \; \le \; C \; \big(2 + \|b^\varepsilon\|_{L_p(B)}\big)^{\mu} \; \Big(\|v^\varepsilon\|_{L_{2p'}(B(\frac34))} + \|f\|_{L_p(B(\frac34))}\Big)$$

Hence $v^\varepsilon$ are equibounded on $B(\frac{1}{2})$. Passing to the limit in the above inequality and taking into account that $v = u$ on $B(\frac{5}{6})$ we obtain

$$\|u\|_{L_\infty(B(\frac{1}{2}))} \;\leq\; C \;\left(2 + \|b\|_{L_p(B)}\right)^\mu \;\left(\|u\|_{L_{2p'}(B(\frac{3}{4}))} + \|f\|_{L_p(B(\frac{3}{4}))}\right)$$

To conclude the proof we remark that for $p > \frac{n}{2}$ due to the imbedding theorem we have

$$\|u\|_{L_{2p'}(B)} \;\leq\; C(n,p)\; \|u\|_{W_2^1(B)}$$

and hence the estimate (1.20) follows. $\square$

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